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QUADRATIC FIELD

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(ii) When f = p (prime) and p does not divide a, we set  $I_1 = I$ . If p divides a, we take for I the ideal  $a_1[1, \phi_1]$  following I in its period. In this case, as  $p \mid a$ , from  $p^2D = b_1^2 + 4aa_1$ , we see that  $p \mid b_1$  and so, as  $GCD(a_1, b_1, a) = 1$  we see that p does not divide  $a_1$ . Then, by (2.12), we have  $I_1 = \rho I$  with  $\rho = \frac{a_1}{a} \phi_1$ . Now, by Proposition 5,  $\phi_1 = \frac{b_1 + \sqrt{D'}}{2a_1}$  is reduced, so that  $1 \leq b_1 < \sqrt{D'}$ , and

$$(7.5) 1 \leqslant a_1 < \sqrt{D'} ,$$

giving

$$(7.6) 1 \leqslant \rho < \sqrt{D'} .$$

The rest of the proof follows exactly as in the proof of (i) using (7.5) (resp. (7.6)) in place of (7.3) (resp. (7.4)).

# 8. Gauss's reduction process

Definition 14. (Half-reduced) A representation  $\{a, b\}$  of an ideal I is said to be half-reduced if

(8.1) 
$$0 < \frac{-b + \sqrt{D}}{2 |c|} < 1,$$

where  $c = (D - b^2) | 4a$ .

An ideal I is called *half-reduced* if there exists a half-reduced representation of I.

Clearly, if  $\{a, b\}$  is half-reduced, then  $b < \sqrt{D}$  and  $\{-a, b\}$  is half-reduced.

LEMMA 7. Let I be a primitive ideal of  $O_D$ . To each representation  $\{a,b\}$  of I corresponds a unique integer q such that the q-neighbour representation  $\{a',b'\}$  is half-reduced. The integer b' and the ideal  $I' = \left[a', \frac{b' + \sqrt{D}}{2}\right]$  are determined by I. The value of q is

(8.2) 
$$q = \frac{a}{|a|} \left[ \frac{b + \sqrt{D}}{2|a|} \right].$$

The representation  $\{a',b'\}$  and the ideal I' are the Gauss neighbour of the representation  $\{a,b\}$  and of the ideal I respectively, so that  $\{a,b\} \xrightarrow{G} \{a',b'\}$ .

*Proof.* As  $c' = \frac{(D - b'^2)}{4a'} = a$  (by (2.10)), the q-neighbour representation  $\{a', b'\}$  of  $\{a, b\}$  is half-reduced if

$$0<\frac{-b'+\sqrt{D}}{2|a|}<1,$$

that is, by (2.10), if  $0 < \frac{b + \sqrt{D}}{2 \mid a \mid} - \frac{a}{\mid a \mid} q < 1$ , giving  $q = \frac{a}{\mid a \mid} \left[ \frac{b + \sqrt{D}}{2 \mid a \mid} \right]$ , which shows that q and  $\{a', b'\}$  are determined by  $\{a, b\}$ . Let  $\{\pm a, b + 2K \mid a \mid\} = \{a_1, b_1\}$  be another representation of I giving rise to a half-reduced representation, say  $\{a'_1, b'_1\}$ . As  $b'_1 \equiv -b_1 \equiv -b \equiv b' \pmod{2 \mid a \mid}$  and  $|a_1| = |a|$ , we see from the inequalities

$$0 < \frac{\sqrt{D} - b'}{2|a|} < 1$$
 and  $0 < \frac{\sqrt{D} - b'_1}{2|a_1|} < 1$ 

that  $b_1' = b'$ . Hence, as  $|a| = |a_1|$  and  $b' = b_1'$ , from  $D = b'^2 + 4aa'$   $= b_1'^2 + 4a_1a_1'$ , we see that  $|a'| = |a_1'|$ . This shows that  $I_1' = I$ , which completes the proof of Lemma 7.

PROPOSITION 11. Let  $\{a,b\}$  be a half-reduced representation of a half-reduced ideal I. Let  $\{a,b\} \xrightarrow{G} \{a',b'\}$  and set  $I' = \left[a',\frac{b'+\sqrt{D}}{2}\right]$ . We have

- (i) if  $b < -\sqrt{D}$  then  $b' > b + 2\sqrt{D}$ ,
- (ii) if  $b > -\sqrt{D}$  then I' is reduced.
- (iii) if I is reduced, then I' is reduced, and moreover if  $\{a,b\}$  is the representation of I such that a>0 and  $\phi=\frac{b+\sqrt{D}}{2a}$  is reduced, then the Lagrange neighbour and the Gauss neighbour are the same.

*Proof.* For any representation  $\{a, b\}$  of any primitive ideal, we have

(8.3) 
$$\left| \frac{\sqrt{D} - b}{2c} \right| \left| \frac{\sqrt{D} + b}{2a} \right| = 1.$$

Now take  $\{a,b\}$  to be a half-reduced representation of the half-reduced ideal I so that  $0 < \frac{-b + \sqrt{D}}{2|c|} < 1$ , where  $c = (D - b^2)/4a$ .

(i) Suppose that  $b < -\sqrt{D}$ . Then we have  $b^2 - D = 4 |a| |c|$  so that (8.3) becomes  $\left(\frac{\sqrt{D} - b}{2 |c|}\right) \left(\frac{-b - \sqrt{D}}{2 |a|}\right) = 1$ . As  $0 < \frac{-b + \sqrt{D}}{2 |c|} < 1$ , we see that  $\frac{-b - \sqrt{D}}{2 |a|} > 1$ . But, as  $\{a', b'\}$  is also half-reduced, we have  $\frac{-b' + \sqrt{D}}{2 |a|} < 1$ , so that  $-b' + \sqrt{D} < 2 |a| < -b - \sqrt{D}$ , proving that  $b' > b + 2\sqrt{D}$ .

(ii) Suppose that  $b > -\sqrt{D}$ . Then, we have  $|b| < \sqrt{D}$ , and (8.3) can be written

$$\left(\frac{\sqrt{D}-b}{2|c|}\right)\left(\frac{\sqrt{D}+b}{2|a|}\right) = 1.$$

showing that  $\frac{\sqrt{D}+b}{2\mid a\mid} > 1$ . Or the other hand, as  $\{a',b'\}$  is half-reduced, we have  $0 < \frac{\sqrt{D}-b'}{2\mid a\mid} < 1$ , that is  $0 < \frac{\sqrt{D}+b}{2\mid a\mid} - \frac{a}{\mid a\mid} q < 1$ , so that

$$\frac{a}{|a|} q = \left[ \frac{\sqrt{D} + b}{2|a|} \right] \geqslant 1.$$

Hence we obtain

$$\sqrt{D} + b' = \sqrt{D} - b + 2aq = (\sqrt{D} - b) + 2|a|\left(\frac{aq}{|a|}\right)| > 2|a|,$$

which, together with the inequalities  $0 < \frac{\sqrt{D} - b'}{2 \mid a \mid} < 1$ , shows that  $\phi'$  is reduced if a > 0 and  $-\phi'$  is reduced if a < 0, proving that I' is reduced. (iii) We suppose that I is reduced and choose the representation  $\{a, b\}$  of I with a > 0 and  $\phi = \frac{b + \sqrt{D}}{2a}$  reduced. As  $\phi$  is half-reduced and  $b > -\sqrt{D}$  from (ii)

we see that I' is reduced. Moreover, the integer q used to obtain both the Lagrange neighbour and the Gauss neighbour of  $\{a, b\}$  is  $[\phi]$ . This shows that the two neighbours of  $\{a, b\}$  are the same and concludes the proof of Proposition 11.

Definition 15. (Gauss's reduction process ([1]: §§ 183-185)) We start with

a primitive ideal  $I_0$  of  $O_D$  and a representation  $\{a, b\}$  of  $I_0$ , and define the sequence of representations  $\{a_n, b_n\}$  of the primitive ideals  $I_n$  by

$$\{a_n,b_n\} \stackrel{G}{\to} \{a_{n+1},b_{n+1}\} \quad (n=0,1,2,\ldots).$$

We now show that Gauss's reduction process leads to a reduced ideal equivalent to  $I_0$ . In addition we give an upper bound for the number of steps required to obtain a reduced ideal  $I_n$  as well as bounds for a quantity  $\rho$  in the relation  $I_n = \rho I_0$ .

Proposition 12. (i) The ideal  $I_n$  is reduced for

$$n > \max\left(\frac{|a_0|}{\sqrt{D}} + 1, 2\right) .$$

(ii) Let I' be the first reduced ideal obtained by applying Gauss's reduction to  $I_0$ . Then  $I = \rho I_0$  with  $\frac{1}{|a_0|} \leq \rho < \sqrt{D}$ .

*Proof.* We suppose that  $n > \max\left(\frac{|a_0|}{\sqrt{D}} + 1, 2\right)$  so that  $n \ge 3$ .

If  $b_1 > -\sqrt{D}$ , by Proposition 11 (ii),  $I_2$  is reduced and so, by Proposition 11 (iii),  $I_n$  is reduced.

Suppose on the other hand that  $b_1 < -\sqrt{D}$  and that  $I_n$  is not reduced. Then, by Proposition 11 (ii), we see that  $b_i < -\sqrt{D}$  for i = 1, 2, ..., n - 1. Then, by Proposition 11 (i), we have

$$b_{n-1} > b_1 + 2(n-2)\sqrt{D}$$
.

Hence we obtain

$$b_{n-1} > -b_0 + 2a_0 \left( \frac{a_0}{|a_0|} \left[ \frac{a_0}{|a_0|} \frac{(b_0 + \sqrt{D})}{2a_0} \right] \right) + 2 \left( \frac{|a_0|}{|\sqrt{D}} - 1 \right) \sqrt{D}$$

$$> -b_0 + 2 |a_0| \left( \frac{b_0 + \sqrt{D}}{2|a_0|} - 1 \right) + 2 \left( \frac{|a_0|}{|\sqrt{D}} - 1 \right) \sqrt{D}$$

$$= -\sqrt{D},$$

which is a contradiction. This completes the proof that  $I_n$  is reduced for  $n > \max\left(\frac{|a_0|}{|\sqrt{D}} + 1, 2\right)$ .

(ii) Let  $I_n$  be the first reduced ideal obtained from  $I_0$  by Gauss's reduction

process. If n = 0 then  $\rho = 1$ , so that  $\frac{1}{|a_0|} \le \rho < \sqrt{D}$ . If  $n \ge 1$  we have  $I_n = \rho I_0$  with (by (2.12))

$$\rho = \left| \frac{a_1}{a_0} \, \phi_1 \dots \frac{a_n}{a_{n-1}} \, \phi_n \, \right| = \left| \frac{a_n}{a_0} \, \middle| \left| \frac{b_1 + \sqrt{D}}{2a_1} \middle| \dots \left| \frac{b_n + \sqrt{D}}{2a_n} \right| \right|.$$

As the representations  $\{a_k, b_k\}$  are half-reduced for  $k \ge 1$ , we see, by (8.3), that  $\left|\frac{b_k + \sqrt{D}}{2a_k}\right| > 1 \ (k \ge 1)$  so that  $\rho > \left|\frac{a_n}{a_0}\right| \ge \frac{1}{|a_0|}$ . On the other hand we have

$$\rho = \left| \frac{b_1 + \sqrt{D}}{2a_0} \right| \dots \left| \frac{b_n + \sqrt{D}}{2a_{n-1}} \right|.$$

As  $\{a_k, b_k\}$  is a half-reduced representation for k = 1, 2, ..., n, we have  $0 < \sqrt{D} - b_k < 2 \mid a_{k-1} \mid$ . Furthermore, for k = 1, 2, ..., n-1, we have  $\sqrt{D} + b_k < 2 \mid a_{k-1} \mid$ , as otherwise  $0 < \sqrt{D} - b_k < 2 \mid a_{k-1} \mid < \sqrt{D} + b_k$ , which is equivalent to  $0 < \sqrt{D} - b_k < 2 \mid a_k \mid < \sqrt{D} + b_k$  so that by (4.2) the primitive ideal  $I_k$  would be reduced. Therefore, for k = 1, 2, ..., n-1, we have

$$|\sqrt{D} + b_k| \le |\sqrt{D} + |b_k| = \begin{cases} |\sqrt{D} + b_k| < 2 |a_{k-1}|, & \text{if } b_k \ge 0, \\ |\sqrt{D} - b_k| < 2 |a_{k-1}|, & \text{if } b_k < 0, \end{cases}$$

so that, as  $\{a_n, b_n\}$  is reduced,

$$\rho < \frac{b_n + \sqrt{D}}{2 \mid a_{n-1} \mid} < \sqrt{D}$$

which completes the proof of Proposition 12.

We remark that Proposition 7 and 12 suggest that Lagrange's reduction process may lead to a reduced ideal much faster than Gauss's reduction process, as the number  $M_0$  of Lemma 6 is much smaller than  $\max\left(\frac{|a_0|}{|\sqrt{D}}+1,2\right)$ .

Example 5. We apply both Lagrange reduction and Gauss reduction to the representation {3655,7068} of the primitive ideal [3655,3534 +  $\sqrt{21}$ ] of  $O_{84}$ . We obtain

$${3655,7068} \xrightarrow{L} {-3417, -7068} \xrightarrow{L} {4,234} \xrightarrow{L} {3,6}$$
 (3 steps)

and

$$\{3655, 7068\} \xrightarrow{G} \{-3417, -7068\} \xrightarrow{G} \{3187, -6600\} \xrightarrow{G} \{-2965, -6148\} \xrightarrow{G} \dots$$
  
 $\xrightarrow{G} \{-1, -12\} \xrightarrow{G} \{-5, 8\} \quad (30 \text{ steps}) .$ 

We remark that  $M_0$  is approximately 8.72 and  $\frac{|a_0|}{\sqrt{D}} + 1$  is approximately 399.8.

## **REFERENCES**

- [1] GAUSS, C.F. Disquisitiones Aritmeticae, in *Untersuchungen über höhere* Arithmetik, reprinted Chelsea Publishing Co., New York (1965).
- [2] LAGRANGE, J.-L. Solution d'un problème d'arithmétique (1766), Œuvres de Lagrange, Vol. 1, pp. 671-731. (Gauthier-Villars (1867)).
- [3] Sur la solution des problèmes indéterminés du second degré (1769), Œuvres de Lagrange, Vol. 2, pp. 377-535. (Gauthier-Villars (1868)).
- [4] LENSTRA, H. W., Jr. On the calculation of regulators and class numbers of quadratic fields. London Math. Soc. Lecture Note Ser. 56 (1982), 123-150.
- [5] SCHOLZ, A. and B. SCHOENEBERG. Einführung in die Zahlentheorie. Walter de Gruyter. Berlin and New York (1973).
- [6] SCHOOF, R.G. Quadratic fields and factorization. *Computational Methods in Number Theory* (H. W. Lenstra Jr. and R. Tijdeman, eds). Math. Centrum Tracts. Number 155, Part II, Amsterdam, 1983, 235-286.
- [7] Shanks, D. The infrastructure of a real quadratic field and its applications. *Proc. 1972. Number Theory Conference*, Boulder, Colorado, 1972, 217-224.
- [8] SMITH, H. J. S. Note on the theory of the Pellian equation, and of binary quadratic forms of a positive determinant. *Proc. Lond. Math. Soc.* 7 (1876), 199-208.
- [9] STEPHENS, A. J. and H. C. WILLIAMS. Some computational aspects on a problem of Eisenstein. Preprint.