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(ii) When  $f = p$  (prime) and  $p$  does not divide  $a$ , we set  $I_1 = I$ . If  $p$  divides  $a$ , we take for  $I$  the ideal  $a_1[1, \phi_1]$  following  $I$  in its period. In this case, as  $p \mid a$ , from  $p^2D = b_1^2 + 4aa_1$ , we see that  $p \mid b_1$  and so, as  $\text{GCD}(a_1, b_1, a) = 1$  we see that  $p$  does not divide  $a_1$ . Then, by (2.12), we have  $I_1 = \rho I$  with  $\rho = \frac{a_1}{a} \phi_1$ . Now, by Proposition 5,  $\phi_1 = \frac{b_1 + \sqrt{D'}}{2a_1}$  is reduced, so that  $1 \leq b_1 < \sqrt{D'}$ , and

$$(7.5) \quad 1 \leq a_1 < \sqrt{D'},$$

giving

$$(7.6) \quad 1 \leq \rho < \sqrt{D'}.$$

The rest of the proof follows exactly as in the proof of (i) using (7.5) (resp. (7.6)) in place of (7.3) (resp. (7.4)).

## 8. GAUSS'S REDUCTION PROCESS

*Definition 14.* (Half-reduced) A representation  $\{a, b\}$  of an ideal  $I$  is said to be *half-reduced* if

$$(8.1) \quad 0 < \frac{-b + \sqrt{D}}{2|c|} < 1,$$

where  $c = (D - b^2) \mid 4a$ .

An ideal  $I$  is called *half-reduced* if there exists a half-reduced representation of  $I$ .

Clearly, if  $\{a, b\}$  is half-reduced, then  $b < \sqrt{D}$  and  $\{-a, b\}$  is half-reduced.

LEMMA 7. Let  $I$  be a primitive ideal of  $O_D$ . To each representation  $\{a, b\}$  of  $I$  corresponds a unique integer  $q$  such that the  $q$ -neighbour representation  $\{a', b'\}$  is half-reduced. The integer  $b'$  and the ideal  $I' = \left[ a', \frac{b' + \sqrt{D}}{2} \right]$  are determined by  $I$ . The value of  $q$  is

$$(8.2) \quad q = \frac{a}{|a|} \left[ \frac{b + \sqrt{D}}{2|a|} \right].$$

The representation  $\{a', b'\}$  and the ideal  $I'$  are the Gauss neighbour of the representation  $\{a, b\}$  and of the ideal  $I$  respectively, so that

$$\{a, b\} \xrightarrow{G} \{a', b'\}.$$

*Proof.* As  $c' = \frac{(D - b'^2)}{4a'} = a$  (by (2.10)), the  $q$ -neighbour representation  $\{a', b'\}$  of  $\{a, b\}$  is half-reduced if

$$0 < \frac{-b' + \sqrt{D}}{2|a|} < 1,$$

that is, by (2.10), if  $0 < \frac{b + \sqrt{D}}{2|a|} - \frac{a}{|a|} q < 1$ , giving  $q = \frac{a}{|a|} \left[ \frac{b + \sqrt{D}}{2|a|} \right]$ , which shows that  $q$  and  $\{a', b'\}$  are determined by  $\{a, b\}$ . Let  $\{\pm a, b + 2K|a|\} = \{a_1, b_1\}$  be another representation of  $I$  giving rise to a half-reduced representation, say  $\{a'_1, b'_1\}$ . As  $b'_1 \equiv -b_1 \equiv -b \equiv b' \pmod{2|a|}$  and  $|a_1| = |a|$ , we see from the inequalities

$$0 < \frac{\sqrt{D} - b'}{2|a|} < 1 \quad \text{and} \quad 0 < \frac{\sqrt{D} - b'_1}{2|a_1|} < 1$$

that  $b'_1 = b'$ . Hence, as  $|a| = |a_1|$  and  $b' = b'_1$ , from  $D = b'^2 + 4aa' = b'^2_1 + 4a_1a'_1$ , we see that  $|a'| = |a'_1|$ . This shows that  $I'_1 = I$ , which completes the proof of Lemma 7.

PROPOSITION 11. Let  $\{a, b\}$  be a half-reduced representation of a half-reduced ideal  $I$ . Let  $\{a, b\} \xrightarrow{G} \{a', b'\}$  and set  $I' = \left[ a', \frac{b' + \sqrt{D}}{2} \right]$ . We have

- (i) if  $b < -\sqrt{D}$  then  $b' > b + 2\sqrt{D}$ ,
- (ii) if  $b > -\sqrt{D}$  then  $I'$  is reduced.
- (iii) if  $I$  is reduced, then  $I'$  is reduced, and moreover if  $\{a, b\}$  is the representation of  $I$  such that  $a > 0$  and  $\phi = \frac{b + \sqrt{D}}{2a}$  is reduced, then the Lagrange neighbour and the Gauss neighbour are the same.

*Proof.* For any representation  $\{a, b\}$  of any primitive ideal, we have

$$(8.3) \quad \left| \frac{\sqrt{D} - b}{2c} \right| \left| \frac{\sqrt{D} + b}{2a} \right| = 1.$$

Now take  $\{a, b\}$  to be a half-reduced representation of the half-reduced ideal  $I$  so that  $0 < \frac{-b + \sqrt{D}}{2|c|} < 1$ , where  $c = (D - b^2)/4a$ .

(i) Suppose that  $b < -\sqrt{D}$ . Then we have  $b^2 - D = 4|a||c|$  so that (8.3) becomes  $\left(\frac{\sqrt{D} - b}{2|c|}\right) \left(\frac{-b - \sqrt{D}}{2|a|}\right) = 1$ . As  $0 < \frac{-b + \sqrt{D}}{2|c|} < 1$ , we see that  $\frac{-b - \sqrt{D}}{2|a|} > 1$ . But, as  $\{a', b'\}$  is also half-reduced, we have  $\frac{-b' + \sqrt{D}}{2|a|} < 1$ , so that  $-b' + \sqrt{D} < 2|a| < -b - \sqrt{D}$ , proving that  $b' > b + 2\sqrt{D}$ .

(ii) Suppose that  $b > -\sqrt{D}$ . Then, we have  $|b| < \sqrt{D}$ , and (8.3) can be written

$$\left(\frac{\sqrt{D} - b}{2|c|}\right) \left(\frac{\sqrt{D} + b}{2|a|}\right) = 1.$$

showing that  $\frac{\sqrt{D} + b}{2|a|} > 1$ . On the other hand, as  $\{a', b'\}$  is half-reduced,

we have  $0 < \frac{\sqrt{D} - b'}{2|a|} < 1$ , that is  $0 < \frac{\sqrt{D} + b}{2|a|} - \frac{a}{|a|}q < 1$ , so that

$$\frac{a}{|a|}q = \left[\frac{\sqrt{D} + b}{2|a|}\right] \geq 1.$$

Hence we obtain

$$\sqrt{D} + b' = \sqrt{D} - b + 2aq = (\sqrt{D} - b) + 2|a| \left(\frac{aq}{|a|}\right) > 2|a|,$$

which, together with the inequalities  $0 < \frac{\sqrt{D} - b'}{2|a|} < 1$ , shows that  $\phi'$  is

reduced if  $a > 0$  and  $-\phi'$  is reduced if  $a < 0$ , proving that  $I'$  is reduced.

(iii) We suppose that  $I$  is reduced and choose the representation  $\{a, b\}$  of  $I$  with  $a > 0$  and  $\phi = \frac{b + \sqrt{D}}{2a}$  reduced. As  $\phi$  is half-reduced and  $b > -\sqrt{D}$  from (ii)

we see that  $I'$  is reduced. Moreover, the integer  $q$  used to obtain both the Lagrange neighbour and the Gauss neighbour of  $\{a, b\}$  is  $[\phi]$ . This shows that the two neighbours of  $\{a, b\}$  are the same and concludes the proof of Proposition 11.

*Definition 15.* (Gauss's reduction process ([1]: §§ 183-185)) We start with

a primitive ideal  $I_0$  of  $O_D$  and a representation  $\{a, b\}$  of  $I_0$ , and define the sequence of representations  $\{a_n, b_n\}$  of the primitive ideals  $I_n$  by

$$\{a_n, b_n\} \xrightarrow{G} \{a_{n+1}, b_{n+1}\} \quad (n = 0, 1, 2, \dots).$$

We now show that Gauss's reduction process leads to a reduced ideal equivalent to  $I_0$ . In addition we give an upper bound for the number of steps required to obtain a reduced ideal  $I_n$  as well as bounds for a quantity  $\rho$  in the relation  $I_n = \rho I_0$ .

PROPOSITION 12. (i) *The ideal  $I_n$  is reduced for*

$$n > \max \left( \frac{|a_0|}{\sqrt{D}} + 1, 2 \right).$$

(ii) *Let  $I'$  be the first reduced ideal obtained by applying Gauss's reduction to  $I_0$ . Then  $I = \rho I_0$  with  $\frac{1}{|a_0|} \leq \rho < \sqrt{D}$ .*

*Proof.* We suppose that  $n > \max \left( \frac{|a_0|}{\sqrt{D}} + 1, 2 \right)$  so that  $n \geq 3$ .

If  $b_1 > -\sqrt{D}$ , by Proposition 11 (ii),  $I_2$  is reduced and so, by Proposition 11 (iii),  $I_n$  is reduced.

Suppose on the other hand that  $b_1 < -\sqrt{D}$  and that  $I_n$  is not reduced. Then, by Proposition 11 (ii), we see that  $b_i < -\sqrt{D}$  for  $i = 1, 2, \dots, n-1$ . Then, by Proposition 11 (i), we have

$$b_{n-1} > b_1 + 2(n-2)\sqrt{D}.$$

Hence we obtain

$$\begin{aligned} b_{n-1} &> -b_0 + 2a_0 \left( \frac{a_0}{|a_0|} \left[ \frac{a_0}{|a_0|} \frac{(b_0 + \sqrt{D})}{2a_0} \right] \right) + 2 \left( \frac{|a_0|}{\sqrt{D}} - 1 \right) \sqrt{D} \\ &> -b_0 + 2|a_0| \left( \frac{b_0 + \sqrt{D}}{2|a_0|} - 1 \right) + 2 \left( \frac{|a_0|}{\sqrt{D}} - 1 \right) \sqrt{D} \\ &= -\sqrt{D}, \end{aligned}$$

which is a contradiction. This completes the proof that  $I_n$  is reduced for  $n > \max \left( \frac{|a_0|}{\sqrt{D}} + 1, 2 \right)$ .

(ii) Let  $I_n$  be the first reduced ideal obtained from  $I_0$  by Gauss's reduction

process. If  $n = 0$  then  $\rho = 1$ , so that  $\frac{1}{|a_0|} \leq \rho < \sqrt{D}$ . If  $n \geq 1$  we have  $I_n = \rho I_0$  with (by (2.12))

$$\rho = \left| \frac{a_1}{a_0} \phi_1 \cdots \frac{a_n}{a_{n-1}} \phi_n \right| = \left| \frac{a_n}{a_0} \right| \left| \frac{b_1 + \sqrt{D}}{2a_1} \right| \cdots \left| \frac{b_n + \sqrt{D}}{2a_n} \right|.$$

As the representations  $\{a_k, b_k\}$  are half-reduced for  $k \geq 1$ , we see, by (8.3), that  $\left| \frac{b_k + \sqrt{D}}{2a_k} \right| > 1$  ( $k \geq 1$ ) so that  $\rho > \left| \frac{a_n}{a_0} \right| \geq \frac{1}{|a_0|}$ . On the other hand we have

$$\rho = \left| \frac{b_1 + \sqrt{D}}{2a_0} \right| \cdots \left| \frac{b_n + \sqrt{D}}{2a_{n-1}} \right|.$$

As  $\{a_k, b_k\}$  is a half-reduced representation for  $k = 1, 2, \dots, n$ , we have  $0 < \sqrt{D} - b_k < 2|a_{k-1}|$ . Furthermore, for  $k = 1, 2, \dots, n-1$ , we have  $\sqrt{D} + b_k < 2|a_{k-1}|$ , as otherwise  $0 < \sqrt{D} - b_k < 2|a_{k-1}| < \sqrt{D} + b_k$ , which is equivalent to  $0 < \sqrt{D} - b_k < 2|a_k| < \sqrt{D} + b_k$  so that by (4.2) the primitive ideal  $I_k$  would be reduced. Therefore, for  $k = 1, 2, \dots, n-1$ , we have

$$|\sqrt{D} + b_k| \leq \sqrt{D} + |b_k| = \begin{cases} \sqrt{D} + b_k < 2|a_{k-1}|, & \text{if } b_k \geq 0, \\ \sqrt{D} - b_k < 2|a_{k-1}|, & \text{if } b_k < 0, \end{cases}$$

so that, as  $\{a_n, b_n\}$  is reduced,

$$\rho < \frac{b_n + \sqrt{D}}{2|a_{n-1}|} < \sqrt{D}$$

which completes the proof of Proposition 12.

We remark that Proposition 7 and 12 suggest that Lagrange's reduction process may lead to a reduced ideal much faster than Gauss's reduction process, as the number  $M_0$  of Lemma 6 is much smaller than  $\max\left(\frac{|a_0|}{\sqrt{D}} + 1, 2\right)$ .

*Example 5.* We apply both Lagrange reduction and Gauss reduction to the representation  $\{3655, 7068\}$  of the primitive ideal  $[3655, 3534 + \sqrt{21}]$  of  $O_{84}$ . We obtain

$$\{3655, 7068\} \xrightarrow{L} \{-3417, -7068\} \xrightarrow{L} \{4, 234\} \xrightarrow{L} \{3, 6\} \quad (3 \text{ steps})$$

and

$$\begin{aligned} \{3655, 7068\} \xrightarrow{G} \{-3417, -7068\} \xrightarrow{G} \{3187, -6600\} \xrightarrow{G} \{-2965, -6148\} \xrightarrow{G} \dots \\ \xrightarrow{G} \{-1, -12\} \xrightarrow{G} \{-5, 8\} \quad (30 \text{ steps}) . \end{aligned}$$

We remark that  $M_0$  is approximately 8.72 and  $\frac{|a_0|}{\sqrt{D}} + 1$  is approximately 399.8.

## REFERENCES

- [1] GAUSS, C.F. *Disquisitiones Arithmeticae*, in *Untersuchungen über höhere Arithmetik*, reprinted Chelsea Publishing Co., New York (1965).
- [2] LAGRANGE, J.-L. Solution d'un problème d'arithmétique (1766), *Œuvres de Lagrange*, Vol. 1, pp. 671-731. (Gauthier-Villars (1867)).
- [3] ——— Sur la solution des problèmes indéterminés du second degré (1769), *Œuvres de Lagrange*, Vol. 2, pp. 377-535. (Gauthier-Villars (1868)).
- [4] LENSTRA, H.W., Jr. On the calculation of regulators and class numbers of quadratic fields. *London Math. Soc. Lecture Note Ser. 56* (1982), 123-150.
- [5] SCHOLZ, A. and B. SCHOENEBERG. *Einführung in die Zahlentheorie*. Walter de Gruyter. Berlin and New York (1973).
- [6] SCHOOF, R.G. Quadratic fields and factorization. *Computational Methods in Number Theory* (H. W. Lenstra Jr. and R. Tijdeman, eds). Math. Centrum Tracts. Number 155, Part II, Amsterdam, 1983, 235-286.
- [7] SHANKS, D. The infrastructure of a real quadratic field and its applications. *Proc. 1972. Number Theory Conference*, Boulder, Colorado, 1972, 217-224.
- [8] SMITH, H.J.S. Note on the theory of the Pellian equation, and of binary quadratic forms of a positive determinant. *Proc. Lond. Math. Soc. 7* (1876), 199-208.
- [9] STEPHENS, A.J. and H.C. WILLIAMS. Some computational aspects on a problem of Eisenstein. Preprint.