

<b>Zeitschrift:</b>	L'Enseignement Mathématique
<b>Herausgeber:</b>	Commission Internationale de l'Enseignement Mathématique
<b>Band:</b>	36 (1990)
<b>Heft:</b>	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
 <b>Artikel:</b>	THE DISTANCE BETWEEN IDEALS IN THE ORDERS OF A REAL QUADRATIC FIELD
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<b>Kapitel:</b>	7. COMPARISON OF DISTANCES BETWEEN CORRESPONDING IDEALS IN DIFFERENT ORDERS
<b>DOI:</b>	<a href="https://doi.org/10.5169/seals-57912">https://doi.org/10.5169/seals-57912</a>

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PROPOSITION 10. *If  $I$  and  $J$  are equivalent, reduced, primitive ideals of  $O_D$  then*

$$d(J, I) \equiv d(I, J)^{-1} \pmod{\times \eta}.$$

*Proof.* As  $I$  and  $J$  are in the same period we have  $J = \rho I (\rho \in K^*)$  and  $I = \sigma J (\sigma \in K^*)$ . As  $I = \rho^{-1}J$  we have  $\sigma \equiv \rho^{-1} \pmod{\times \eta}$ , which proves Proposition 10.

## 7. COMPARISON OF DISTANCES BETWEEN CORRESPONDING IDEALS IN DIFFERENT ORDERS

Let  $C$  be a primitive class of the order  $O_{Df^2}$  and let  $\theta(C)$  be the image of  $C$  by the mapping  $\theta$  defined in §3. As an application of the concept of distance described in §6, we explain how to define a mapping of the period of  $C$  into the period of  $\theta(C)$ , which approximately preserves distance.

**THEOREM 2.** *For  $D' = Df^2$  let  $C \in C_{D'}$  and  $\theta(C)$  its image by the surjective homomorphism  $\theta: C_{D'} \rightarrow C_D$ .*

(i) *There exists a mapping  $\tau$  from the period of  $C$  into the period of  $\theta(C)$  such that for  $I$  and  $I'$  in the period of  $C$  we have, for a choice of  $d$  modulo units,*

$$(7.1) \quad \frac{d(I, I')}{8f^7 D^{3/2}} < d(\tau(I), \tau(I')) < 8f^7 D^{3/2} d(I, I').$$

(ii) *When  $f = p$  (prime) there exists a mapping  $\sigma$  from the period of  $C$  into the period of  $\theta(C)$  such that for  $I$  and  $I'$  in the period of  $C$  we have, for a choice  $d$  modulo units,*

$$(7.2) \quad \frac{d(I, I')}{2Dp^2} < d(\sigma(I), \sigma(I')) < 2Dp^2 d(I, I').$$

*Proof.* Let  $I = a[1, \phi] (a > 0)$  and  $I' = a'[1, \phi'] (a' > 0)$  be two equivalent, reduced, primitive ideals of a class  $C$  of  $O_{D'} (D' = Df^2)$  with  $\phi = \frac{b + \sqrt{D'}}{2a}$  and  $\phi' = \frac{b' + \sqrt{D'}}{2a'}$  reduced. Let  $\delta \in K^*$  be such that  $I' = \delta I, \delta > 0$ .

(i) If  $GCD(a, f) = 1$  we set  $I_1 = I$ . If  $GCD(a, f) > 1$ , from the proof of Lemma 2, we see that there exists an ideal  $I_1 = a_1[1, \phi_1] = \rho I$  in  $C$  with

$\rho = |x + \bar{\phi}y|$ , where  $x$  and  $y$  are integers such that  $a_1 = |ax^2 + bxy - \left(\frac{D' - b^2}{4a}\right)y^2|$ ,  $GCD(a_1, f) = 1$ ,  $GCD(x, y) = 1$ ,  $0 \leq x < f$ ,  $0 \leq y < f$ .

As  $\phi = \frac{b + \sqrt{D'}}{2a}$  is reduced, we have

$$1 \leq a < \sqrt{D'}, \quad 1 \leq b < \sqrt{D'}, \quad 1 \leq c < \sqrt{D'} \quad \left( c = \frac{D' - b^2}{4a} \right),$$

so that  $\phi < \sqrt{D'}$ ,  $|\bar{\rho}| = x + \bar{\phi}y < f(1 + \sqrt{D'}) < 2f\sqrt{D'}$ ,

and

$$(7.3) \quad 1 \leq a_1 < 2\sqrt{D'} f^2.$$

Also  $\phi > 1$ ,  $-1 < \bar{\phi} < 0$ , so, as  $\rho |\bar{\rho}| = a_1/a$ , we have

$$(7.4) \quad \frac{1}{2fD'} < \rho < f.$$

By the way in which we have defined  $I_1 = \left[ a_1, \frac{b_1 + \sqrt{D'}}{2} \right]$ , we have

$GCD(a_1, f) = 1$ . Appealing to the proof of Theorem 1 (i), we see that there exists an integer  $b_2$  such that  $I_1 = \left[ a_1, f \left( \frac{b_2 + \sqrt{D}}{2} \right) \right]$ .

Similarly there exists an ideal  $I'_1 = \left[ a'_1, f \left( \frac{b'_2 + \sqrt{D}}{2} \right) \right]$  such that  $I'_1 = \rho' I'$

with  $\rho'$  satisfying (7.4). Now, by Theorem 1,  $J_1 = \left[ a_1, \frac{b_2 + \sqrt{D}}{2} \right]$  and

$J'_1 = \left[ a'_1, \frac{b'_2 + \sqrt{D}}{2} \right]$  are ideals of  $\theta(C)$  such that  $J'_1 = \rho' \delta \rho^{-1} J_1$ . Applying the Lagrange reduction process to  $J_1$  and  $J'_1$ , we obtain reduced ideals  $J$  and  $J'$ , and, by Proposition 7, we have  $J = \alpha J_1$ , and  $J' = \alpha' J'_1$ , with (by (7.3))

$$\frac{1}{2f^2\sqrt{D'}} < \frac{1}{a_1} \leq \alpha < 2, \quad \frac{1}{2f^2\sqrt{D'}} < \frac{1}{a'_1} \leq \alpha' < 2.$$

Thus we have  $J' = \delta' J$ , where  $\delta' = \alpha' \rho' \delta \rho^{-1} \alpha^{-1}$  satisfies

$$\frac{\delta}{8f^4 D'^{3/2}} < \delta' < 8f^4 D'^{3/2} \delta.$$

Setting  $J = \tau(I)$  gives the required mapping and proves (7.1).

(ii) When  $f = p$  (prime) and  $p$  does not divide  $a$ , we set  $I_1 = I$ . If  $p$  divides  $a$ , we take for  $I$  the ideal  $a_1[1, \phi_1]$  following  $I$  in its period. In this case, as  $p | a$ , from  $p^2D = b_1^2 + 4aa_1$ , we see that  $p | b_1$  and so, as  $\text{GCD}(a_1, b_1, a) = 1$  we see that  $p$  does not divide  $a_1$ . Then, by (2.12), we have

$I_1 = \rho I$  with  $\rho = \frac{a_1}{a} \phi_1$ . Now, by Proposition 5,  $\phi_1 = \frac{b_1 + \sqrt{D'}}{2a_1}$  is reduced, so that  $1 \leq b_1 < \sqrt{D'}$ , and

$$(7.5) \quad 1 \leq a_1 < \sqrt{D'} ,$$

giving

$$(7.6) \quad 1 \leq \rho < \sqrt{D'} .$$

The rest of the proof follows exactly as in the proof of (i) using (7.5) (resp. (7.6)) in place of (7.3) (resp. (7.4)).

## 8. GAUSS'S REDUCTION PROCESS

*Definition 14.* (Half-reduced) A representation  $\{a, b\}$  of an ideal  $I$  is said to be *half-reduced* if

$$(8.1) \quad 0 < \frac{-b + \sqrt{D}}{2|c|} < 1 ,$$

where  $c = (D - b^2) | 4a$ .

An ideal  $I$  is called *half-reduced* if there exists a half-reduced representation of  $I$ .

Clearly, if  $\{a, b\}$  is half-reduced, then  $b < \sqrt{D}$  and  $\{-a, b\}$  is half-reduced.

**LEMMA 7.** Let  $I$  be a primitive ideal of  $O_D$ . To each representation  $\{a, b\}$  of  $I$  corresponds a unique integer  $q$  such that the  $q$ -neighbour representation  $\{a', b'\}$  is half-reduced. The integer  $b'$  and the ideal  $I' = \left[ a', \frac{b' + \sqrt{D}}{2} \right]$  are determined by  $I$ . The value of  $q$  is

$$(8.2) \quad q = \frac{a}{|a|} \left[ \frac{b + \sqrt{D}}{2|a|} \right] .$$