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$< \frac{2a_{n_0-1}}{a_{n_0}}$ . Then, appealing to (5.20), we obtain

$$1 < \phi_1 \dots \phi_{n_0} < \frac{2a_0}{a_{n_0}B_{n_0-3}},$$

so that, by (5.17), we have

$$\frac{a_{n_0}}{a_0} < \delta < \frac{2}{B_{n_0-3}}.$$

It remains to consider the case  $n_0 = 1$ . If  $I_0$  is reduced then  $\delta = 1$ . If  $I_0$  is not reduced then  $\delta = \frac{a_1}{a_0} \phi_1$  and, as above, we have  $1 < \phi_1 < \frac{2a_0}{a_1}$ , giving

$$\frac{a_1}{a_0} < \delta < 2.$$

Hence in all cases we have  $\frac{1}{a_0} \leq \delta < 2$ . All subsequent Lagrange neighbours of  $I$  are reduced by Lemma 5. This completes the proof of Proposition 7.

## 6. PERIODS OF REDUCED CYCLES

We show that any two equivalent reduced, primitive ideals of the same order  $O_D$  can be obtained from one another by using the Lagrange reduction process described in §5.

**PROPOSITION 8.** ([5]: §31, [12]: Theorem 4.5) *Let  $I = a[1, \phi]$  ( $a > 0$ ) and  $J = b[1, \psi]$  ( $b > 0$ ) be two equivalent, reduced, primitive ideals of  $O_D$ , so that  $[1, \psi] = \rho[1, \phi]$  for some  $\rho(> 0) \in K^*$ . Interchanging  $I$  and  $J$  if necessary we may suppose that  $\rho \geq 1$ . Set  $I_0 = I$ . Then there exists a non negative integer  $n$  such that  $J = I_n$  and  $\rho = \phi_1 \dots \phi_n$ , so that  $J = I_n = \rho_n I$ .*

*Proof.* Recalling that  $\phi_n > 1$  ( $n \geq 1$ ), we see from (5.10) and (5.13) that the sequence  $\{\phi_1 \dots \phi_n\}_{n=0}^{\infty}$  is monotonically increasing and unbounded. Hence there exists an integer  $n \geq 0$  such that  $\phi_1 \dots \phi_n \leq \rho < \phi_1 \dots \phi_{n+1}$ . As

$I_n = \frac{a_n}{a_0} \phi_1 \dots \phi_n I_0$  (by (5.5)), we have  $\frac{1}{b} J = \frac{\rho}{\phi_1 \dots \phi_n} \frac{1}{a_n} I_n$ . If  $\rho = \phi_1 \dots \phi_n$  then

$\frac{1}{b}J = \frac{1}{a_n}I_n$  and so, by Proposition 2 (iii), we have  $b = a_n$  and  $J = I_n$  as required. This we may suppose that  $\rho > \phi_1 \dots \phi_n$ . Replacing  $I_0$  by  $I_n$ , we obtain

$$(6.1) \quad \frac{1}{b}J = \rho \frac{1}{a_0}I_0, \quad \text{where } 1 < \rho < \phi_1.$$

From (6.1), we see that  $\frac{a_0}{\rho}J = bI_0$ , and so, as  $J\bar{J} = (b)$ , we have  $\frac{a_0}{\rho} = I_0\bar{J}$ ,

showing that  $\frac{1}{\rho} \in \frac{1}{a_0}I_0$ . Next we observe that

$$\frac{1}{a_0}I_0 = \frac{1}{\phi_1 a_1}I_1 = \frac{1}{\phi_1}[1, \phi_1] = \left[1, \frac{1}{\phi_1}\right],$$

so there are integers  $x$  and  $y$  such that

$$\frac{1}{\rho} = x + \frac{y}{\phi_1}.$$

Thus, as  $1 < \rho < \phi_1$ , we have

$$(6.2) \quad \frac{1}{\phi_1} < x + \frac{y}{\phi_1} < 1.$$

Appealing to (6.1), we obtain

$$J = \frac{b\rho}{a_0}I_0 = \frac{b\rho}{a_1\phi_1}I_1 = \frac{b\rho}{\phi_1}[1, \phi_1],$$

so that  $\frac{b\rho}{\phi_1} \in J$ , and  $0 < \frac{b\rho}{\phi_1} < b$ . As  $J$  is reduced, by Proposition 4, we have

$$\left| \frac{b\rho}{\phi_1} \right| = \frac{b|\rho|}{|\phi_1|} > b, \quad \text{so that } \left| \frac{1}{\rho} \right| < \left| \frac{1}{\phi_1} \right|, \quad \text{that is}$$

$$(6.3) \quad \left| x + \frac{y}{\phi_1} \right| < \frac{1}{|\phi_1|}.$$

From (6.2) we see that  $y \neq 0$ . Then (6.3) shows that  $x \neq 0$ , and that, as  $\bar{\phi}_1 < 0$ ,  $xy > 0$ . This contradicts (6.2), and completes the proof of Proposition 8.

Let  $I_0$  be a reduced, primitive ideal of a class  $C$  of  $O_D$ . By the Lagrange reduction process described in §5, we obtain (by Proposition 5) an infinite

sequence  $\{I_n\}_{n=0}^{\infty}$  of reduced, primitive ideals with each ideal  $I_n$  equivalent to  $I_0$ . By Proposition 8, this sequence contains all the reduced, primitive ideals of the class  $C$ . As  $C$  contains only a finite number of reduced, primitive ideals (§4), there exist integers  $r$  and  $l$  with  $0 \leq r < r+l$  such that  $I_r = I_{r+l}$ . Applying Proposition 6 (ii), we obtain successively  $I_{r-1} = I_{r+l-1}$ ,  $I_{r-2} = I_{r+l-2}$ , ..., and, after  $r$  steps, we have  $I_0 = I_l$ , which shows that the sequence  $\{I_n\}_{n=0}^{\infty}$  is purely periodic.

*Definition 12.* (Period) Let  $I_0$  be a reduced, primitive ideal of a class  $C$  of  $O_D$ . Let  $l$  be the least positive integer with  $I_0 = I_l$ . The set  $\{I_0, \dots, I_{l-1}\}$  is called the *period* of the class  $C$ . The length of the period is the integer  $l$ .

The period of the class  $C$  of  $O_D$  consists of all the reduced, primitive ideals in  $C$ . It is easy to see that if  $I_s = I_t$  then  $l$  divides  $s - t$ . As  $I_l = I_0$ , we see, from (5.5), that  $I_0 = \eta I_0$ , where

$$(6.4) \quad \eta = \rho_l = \prod_{i=1}^l \phi_i,$$

and so, by Proposition 2 (ii),  $\eta$  is a unit ( $> 1$ ) of  $O_D$ .

PROPOSITION 9. (i) If  $I = I_0$  and  $J$  are equivalent, reduced, primitive ideals of  $O_D$  with  $J = \alpha I_0$ , where  $\alpha (\geq 1) \in K^*$ , then there exist unique integers  $q$  and  $s$  such that

$$\alpha = \eta^q \rho_s \quad (\rho_s \text{ is defined in (5.5), } \eta \text{ in (6.4)})$$

where

$$q \geq 0, \quad 0 \leq s \leq l-1.$$

(ii) If  $J = I$  then we have  $s = 0$  and  $\alpha = \eta^q$ .

*Proof.* (i) By Proposition 8 there exists a nonnegative integer  $n$  such that

$$J = I_n = \rho_n I_0, \quad \alpha = \rho_n.$$

Let  $q (\geq 0)$  and  $s$  be the integers defined uniquely by

$$n = ql + s, \quad 0 \leq s \leq l-1.$$

Then, by periodicity, we have

$$\alpha = \rho_s (\rho_l)^q = \eta^q \rho_s,$$

where

$$\eta = \rho_l = \phi_1 \dots \phi_l.$$

This shows the existence of the integers  $q (\geq 0)$  and  $s (0 \leq s \leq l-1)$ .

We next show that  $q$  and  $s$  are unique. Suppose we have  $\alpha = \eta^{q_1} \rho_{s_1} = \eta^{q_2} \rho_{s_2}$  with  $s_1 \leq s_2$ . If  $s_2 > s_1$  then  $q_1 > q_2$  and, appealing to (5.5) and recalling that  $-1 < \phi_i < 0 (i \geq 1)$ , we obtain

$$\eta \leq \eta^{q_1 - q_2} = \frac{\rho_{s_2}}{\rho_{s_1}} = \prod_{i=s_1+1}^{s_2} \left( \frac{-1}{\phi_i} \right) < \prod_{i=1}^l \left( \frac{-1}{\phi_i} \right) = \eta,$$

which is a contradiction. Hence we must have  $s_1 = s_2$ . Then  $\eta^{q_1} = \eta^{q_2}$  and, as  $\eta > 1$ , we must have  $q_1 = q_2$ . This completes the proof of (i).

(ii) From the proof of (i) we see that  $I_n = J = I_0$ , so that  $l \mid n$ , and thus  $q = n/l$  and  $s = 0$ .

**COROLLARY 5.**  $\eta = \prod_{i=1}^l \phi_i$  is a unit ( $> 1$ ) of  $O_D$  such that every unit  $\varepsilon$  of  $O_D$  is given by  $\varepsilon = \pm \eta^r$ , where  $r$  is an integer.  $\eta$  is called the fundamental unit of  $O_D$ .

*Proof.* Let  $\varepsilon$  be a unit of  $O_D$  and let

$$\delta = \begin{cases} \varepsilon, & \text{if } \varepsilon \geq 1, \\ 1/\varepsilon, & \text{if } 0 < \varepsilon < 1, \\ -1/\varepsilon, & \text{if } -1 < \varepsilon < 0, \\ -\varepsilon, & \text{if } \varepsilon \leq -1, \end{cases}$$

so that  $\delta$  is a unit of  $O_D$  satisfying  $\delta \geq 1$ . Applying Proposition 9 (ii) to  $I_0$  and  $J = \delta I_0$ , we see that  $\delta = \eta^q$ , and so  $\varepsilon = \pm \eta^r$ .

Corollary 5 was first proved by Lagrange in the case of the principal class [3: p. 452] (see also [8]). We see that the theory of periods of reduced, primitive ideals in  $O_D$  not only gives the structure of the group of units of  $O_D$  but also provides the structure of each period (the "infrastructure" of Shanks [7]).

**COROLLARY 6.** With  $I_0$  a reduced, primitive ideal of  $O_D$ , we have

$$(i) \quad \eta = B_{l-1} \phi_0 + B_{l-2},$$

$$(ii) \quad \eta = A_{l-1} - B_{l-1} \bar{\phi}_0,$$

$$(iii) \quad l \log \left( \frac{1 + \sqrt{5}}{2} \right) \leq \log \eta < l \log \sqrt{D}$$

*Proof.* Taking  $n = Nl(N = 1, 2, \dots)$  in (5.13) we obtain, as  $\phi_{Nl} = \phi_0$ ,

$$(6.5) \quad \eta^N = B_{Nl-1}\phi_0 + B_{Nl-2}.$$

The assertion (i) is the case  $N = 1$ .

From (5.7), (5.9) and (5.13), we obtain for  $n \geq 1$

$$\phi_1 \dots \phi_n = \frac{(-1)^{n-1}}{B_{n-1}\phi_0 - A_{n-1}}.$$

Taking  $n = Nl(N = 1, 2, \dots)$  and recalling that  $\eta\bar{\eta} = (-1)^l$ , we obtain

$$\eta^N = -\frac{(\eta\bar{\eta})^N}{B_{Nl-1}\phi_0 - A_{Nl-1}}, \text{ so that taking conjugates we deduce}$$

$$(6.6) \quad \eta^N = A_{Nl-1} - B_{Nl-1}\bar{\phi}_0.$$

The assertion (ii) is the case  $N = 1$ .

From (6.5) and (5.10) we have

$$\eta^N > B_{Nl-1} + B_{Nl-2} \geq \left(\frac{1+\sqrt{5}}{2}\right)^{Nl-2} + \left(\frac{1+\sqrt{5}}{2}\right)^{Nl-3} = \left(\frac{1+\sqrt{5}}{2}\right)^{Nl-1},$$

so that

$$\eta > \left(\frac{1+\sqrt{5}}{2}\right)^{l-(1/N)} \quad (N = 1, 2, 3, \dots).$$

Letting  $N \rightarrow \infty$ , we obtain

$$\eta \geq \left(\frac{1+\sqrt{5}}{2}\right)^l,$$

proving the first equality in (iii).

Finally, as  $\phi_i < \sqrt{D}(i \geq 0)$ , we have

$$\eta = \phi_1 \dots \phi_l < (\sqrt{D})^l,$$

proving the second assertion in (iii).

*Example 3.* ( $D = 1892$ ) The period of the class containing the ideal  $[1, 21 + \sqrt{473}]$  is

$$\{[1, 21 + \sqrt{473}], [32, 21 + \sqrt{473}], [11, 11 + \sqrt{473}], [32, 11 + \sqrt{473}]\}.$$

Thus, by Corollary 5, the fundamental unit of  $O_{1892}$  is

$$(21 + \sqrt{473}) \left(\frac{21 + \sqrt{473}}{32}\right) \left(\frac{11 + \sqrt{473}}{11}\right) \left(\frac{11 + \sqrt{473}}{32}\right)$$

$$\begin{aligned}
&= \frac{1}{11 \cdot 32^2} (21 + \sqrt{473})^2 (11 + \sqrt{473})^2 \\
&= \frac{1}{11 \cdot 32^2} (704 + 32\sqrt{473})^2 \\
&= \frac{1}{11} (22 + \sqrt{473})^2 \\
&= 87 + 4\sqrt{473} \\
&= 87 + 2\sqrt{1892}.
\end{aligned}$$

The period of the class containing the ideal  $[7, 16 + \sqrt{473}]$  is

$$\begin{aligned}
&\{[7, 16 + \sqrt{473}], [16, 19 + \sqrt{473}], [19, 13 + \sqrt{473}], [23, 6 + \sqrt{473}], \\
&\quad [8, 17 + \sqrt{473}], [31, 15 + \sqrt{473}]\}
\end{aligned}$$

so, by Corollary 5, the fundamental unit of  $O_{1892}$  is also given by

$$\begin{aligned}
&\left(\frac{16 + \sqrt{473}}{7}\right) \left(\frac{19 + \sqrt{473}}{16}\right) \left(\frac{13 + \sqrt{473}}{19}\right) \left(\frac{6 + \sqrt{473}}{23}\right) \left(\frac{17 + \sqrt{473}}{8}\right) \left(\frac{15 + \sqrt{473}}{31}\right) \\
&= \left(\frac{111 + 5\sqrt{473}}{16}\right) \left(\frac{29 + \sqrt{473}}{23}\right) \left(\frac{91 + 4\sqrt{473}}{31}\right) \\
&= \frac{(349 + 16\sqrt{473})}{23} \frac{(91 + 4\sqrt{473})}{31} \\
&= 87 + 4\sqrt{473} = 87 + 2\sqrt{1892}.
\end{aligned}$$

We are now in a position to define the distance between two reduced, primitive ideals in the same period.

*Definition 13.* (Distance between ideals) If  $I$  and  $J$  are equivalent, reduced, primitive ideals of  $O_D$  then we define the (multiplicative) distance  $d(I, J)$  from  $I$  to  $J$  by

$$d(I, J) \equiv \rho_s(\text{mod } \times \eta)$$

where  $\rho_s$  is given as in Proposition 9 (i).

It is clear that  $d(I, I) = 1$ .

*Example 4.* ( $D = 1892$ ) The two reduced, primitive ideals

$$I = [19, 6 + \sqrt{473}] \quad \text{and} \quad J = [31, 16 + \sqrt{473}]$$

of  $O_{1892}$  are equivalent. Applying the Lagrange reduction process to  $[19, 6 + \sqrt{473}]$ , we obtain

$$[19, 6 + \sqrt{473}] \xrightarrow{L} [16, 13 + \sqrt{473}] \xrightarrow{L} [7, 19 + \sqrt{473}] \xrightarrow{L} [31, 16 + \sqrt{473}] ,$$

so that

$$\begin{aligned} d(I, J) = \rho_3 &= \frac{31}{19} \left( \frac{13 + \sqrt{473}}{16} \right) \left( \frac{19 + \sqrt{473}}{7} \right) \left( \frac{16 + \sqrt{473}}{31} \right) \\ &= \frac{(13 + \sqrt{473}) (111 + 5\sqrt{473})}{19 \times 16} \\ &= \frac{238 + 11\sqrt{473}}{19} . \end{aligned}$$

On the other hand, applying the Lagrange reduction process to  $[31, 16 + \sqrt{473}]$ , we obtain

$$[31, 16 + \sqrt{473}] \xrightarrow{L} [8, 15 + \sqrt{473}] \xrightarrow{L} [23, 17 + \sqrt{473}] \xrightarrow{L} [19, 6 + \sqrt{473}] ,$$

so that

$$\begin{aligned} d(J, I) &= \frac{19}{31} \left( \frac{15 + \sqrt{473}}{8} \right) \left( \frac{17 + \sqrt{473}}{23} \right) \left( \frac{6 + \sqrt{473}}{19} \right) \\ &= \frac{(91 + 4\sqrt{473}) (6 + \sqrt{473})}{31 \times 23} \\ &= \frac{2438 + 115\sqrt{473}}{31 \times 23} \\ &= \frac{106 + 5\sqrt{473}}{31} . \end{aligned}$$

We note that

$$\begin{aligned} &\left( \frac{238 + 11\sqrt{473}}{19} \right) \left( \frac{106 + 5\sqrt{473}}{31} \right) \\ &= \frac{51243 + 2356\sqrt{473}}{589} \\ &= 87 + 4\sqrt{473} = \eta \\ &\equiv 1 \pmod{\times \eta} . \end{aligned}$$



PROPOSITION 10. *If  $I$  and  $J$  are equivalent, reduced, primitive ideals of  $O_D$  then*

$$d(J, I) \equiv d(I, J)^{-1} \pmod{\times \eta}.$$

*Proof.* As  $I$  and  $J$  are in the same period we have  $J = \rho I (\rho \in K^*)$  and  $I = \sigma J (\sigma \in K^*)$ . As  $I = \rho^{-1} J$  we have  $\sigma \equiv \rho^{-1} \pmod{\times \eta}$ , which proves Proposition 10.

## 7. COMPARISON OF DISTANCES BETWEEN CORRESPONDING IDEALS IN DIFFERENT ORDERS

Let  $C$  be a primitive class of the order  $O_{Df^2}$  and let  $\theta(C)$  be the image of  $C$  by the mapping  $\theta$  defined in §3. As an application of the concept of distance described in §6, we explain how to define a mapping of the period of  $C$  into the period of  $\theta(C)$ , which approximately preserves distance.

THEOREM 2. *For  $D' = Df^2$  let  $C \in C_{D'}$  and  $\theta(C)$  its image by the surjective homomorphism  $\theta: C_{D'} \rightarrow C_D$ .*

(i) *There exists a mapping  $\tau$  from the period of  $C$  into the period of  $\theta(C)$  such that for  $I$  and  $I'$  in the period of  $C$  we have, for a choice of  $d$  modulo units,*

$$(7.1) \quad \frac{d(I, I')}{8f^7 D^{3/2}} < d(\tau(I), \tau(I')) < 8f^7 D^{3/2} d(I, I').$$

(ii) *When  $f = p$  (prime) there exists a mapping  $\sigma$  from the period of  $C$  into the period of  $\theta(C)$  such that for  $I$  and  $I'$  in the period of  $C$  we have, for a choice  $d$  modulo units,*

$$(7.2) \quad \frac{d(I, I')}{2Dp^2} < d(\sigma(I), \sigma(I')) < 2Dp^2 d(I, I').$$

*Proof.* Let  $I = a[1, \phi] (a > 0)$  and  $I' = a'[1, \phi'] (a' > 0)$  be two equivalent, reduced, primitive ideals of a class  $C$  of  $O_{D'} (D' = Df^2)$  with  $\phi = \frac{b + \sqrt{D'}}{2a}$  and  $\phi' = \frac{b' + \sqrt{D'}}{2a'}$  reduced. Let  $\delta \in K^*$  be such that  $I' = \delta I, \delta > 0$ .

(i) If  $\text{GCD}(a, f) = 1$  we set  $I_1 = I$ . If  $\text{GCD}(a, f) > 1$ , from the proof of Lemma 2, we see that there exists an ideal  $I_1 = a_1[1, \phi_1] = \rho I$  in  $C$  with