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III. A SKEIN MODEL FOR THE HOMFLY POLYNOMIAL

In this section K is an *oriented* diagram with underlying universe U. A template T is given for U.

Consider the template skein algorithm of section 2. At each stage in the algorithm an active crossing produces (oriented case) a spliced diagram and a diagram with a switched crossing. It is easy to see that *once a crossing has been switched by the template skein algorithm it remains unchanged in all diagrams in the tree that are descendants of the switched diagram.* (The crossing becomes inactive for all the descendants.) Consequently, we may encircle such a crossing to indicate its inactivity, as was done in Figure 5.

By the same token, consider a crossing that has been spliced from an active crossing by the template skein algorithm. As shown in Figures 5 and 6, if the splice occurs from a positive crossing, then in the splice — *the first passage from basepoint goes through the right hand (or lower) strand of the splice.* If the splice occurs from a negative crossing, then in the splice — *the first passage from basepoint goes through the left hand (or upper) strand of the splice.* Here right and left refer to an observer who stands between the strands of the splice, facing in the direction of the orientation. First passage is indicated in Figures 5 and 6 by a dot on the first passage strand, coupled by a wavy line between the strands of the splice. *This location of the first passage strand is inherited by all the descendants of the given diagram.* This property follows from the precedence structure of the template skein algorithm. (Previously spliced and switched crossings are visited first by the algorithm, before it finds the next activity.)

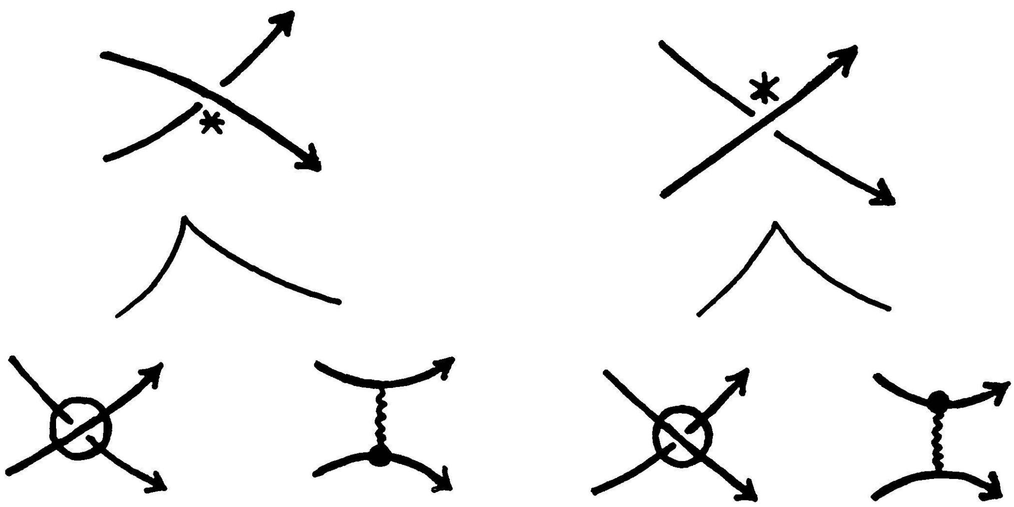


FIGURE 6

As a result of these two remarks we see that *each end node in the tree produced by the template skein algorithm is unique.* That is there are no repetitions among the unlinks produced by the algorithm. Furthermore, each end-node has each of its spliced sites decorated either by a dot and wavy line — indicating first passage on the right or on the left. *This first passage refers to the action of the template skein algorithm applied to the given end-node.* In each case, first passage on the right corresponds to a positive crossing in the original link K, while first passage on the left corresponds to a negative crossing in K.

Definition 3.1. Let K be a given link diagram. Let U be the universe for K, and T a template for U. Let L be a standard unlink relative to T (see Definition 2.3). L is said to be *admissible with respect to* K if at each splice in L, the first passage of the template skein algorithm is on the right hand strand for positive crossings of K and on the left hand strand for negative crossings of K.

The preceding remarks may be summarized by the proposition.

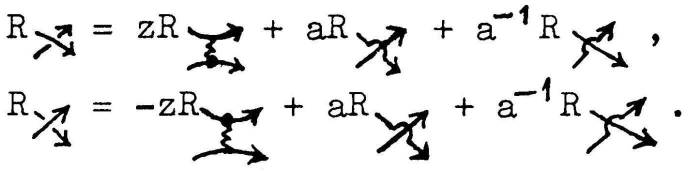
PROPOSITION 3.2. *Let K be a link diagram with universe U, and a given template T for U. Then the admissible unlinks relative to K are in one-to-one correspondence with the end-nodes of the template skein algorithm applied to K (using the template T).*

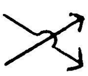
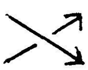
We can now write the state summation

$$R_K = \sum (-1)^{t_-(L)} z^{t(L)} a^{w(L)} \delta^{|L|-1},$$

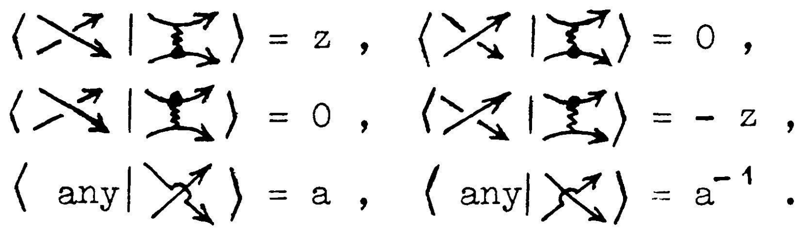
summation over $L \in A(K, T)$, where $A(K, T)$ denotes the set of admissible unlinks relative to T. As in section 2, $t_-(L)$ denotes the number of splices of negative crossings in K, $t(L)$ denotes the number of splices of crossings in K needed to obtain L. $w(L)$ denotes the writhe of L, $|L|$ denotes the number of components of L, and $\delta = (a-a^{-1})/z$.

This state summation can be expressed symbolically by the equations:



Here the symbol  means the same as  except that I use this notation to differentiate a state in $A(K, T)$ from the original link K. Expansion via these equations leads to a large collection of state diagrams many of them inadmissible. However, an inadmissible diagram has some of its vertex weights equal to zero:

$$R_K = \sum_{L \in D(K,\,T)} \langle K \mid L \rangle\, \delta^{|L|-1}$$



Here $D(K, T)$ denotes a set of diagrammatic states (obtained by replacing crossings of K by one of the four local glyphs indicated by the expansion equations). Each state in $D(K, T)$ must satisfy:

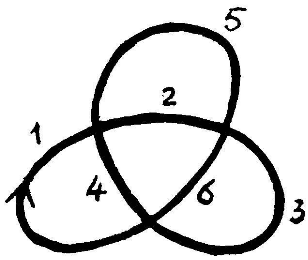
1. Dotted segments must denote first passage relative to the template T.

2. Over and under-crossing indications () must indicate first passage relative to the template T. Notations such as

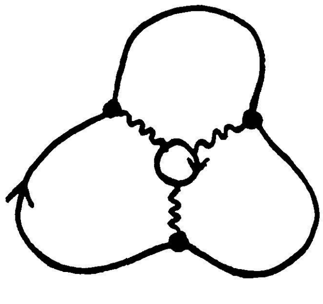
denote the local vertex weights. $\langle K \mid L \rangle$ denotes the product of vertex weights from the sites of the diagrammatic state L.

Note that $D(K, T)$ and $A(K, T)$ are in one-to-one correspondence. Hence this diagrammatic description of the model is equivalent to the summation involving the admissible unlinks.

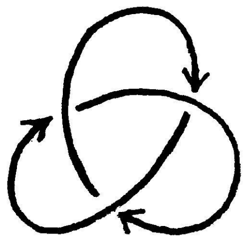
In these expansion formulas the writhe contribution is picked up by the coefficients a and a^{-1}. The decorations and over-passes on an admissible state must be compatible with the template and circuit structure. Thus for the template



the state



is admissible and contributes $(-z)^3\delta$ to the summation for the negative trefoil



IV. A Skein Model for the Kauffman polynomial

The work of section 3 goes over essentially verbatim for the Dubrovnik version of the Kauffman polynomial. Recall that in this context a template is obtained by first orienting the edges of the universe U underlying the unoriented K, and then labelling the edges of U.

The chart in Figure 7 shows the cases of admissible splices at crossings (with respect to the skein template algorithm). Each splice has been labelled with its corresponding vertex weight. Note that a splice is admissible if it indicates the form of passage that is obtained from an approach to the crossing that meets it as an under-crossing. Such approaches give active crossings in the skein template algorithm.

I have retained only the arrow for the first passage after each split, because the orientation on the other edge may change under the direction of the template. *The crossings are oriented because each end-node (unlink) produced by the skein template algorithm acquires an orientation from the directions of travel given by the template.*

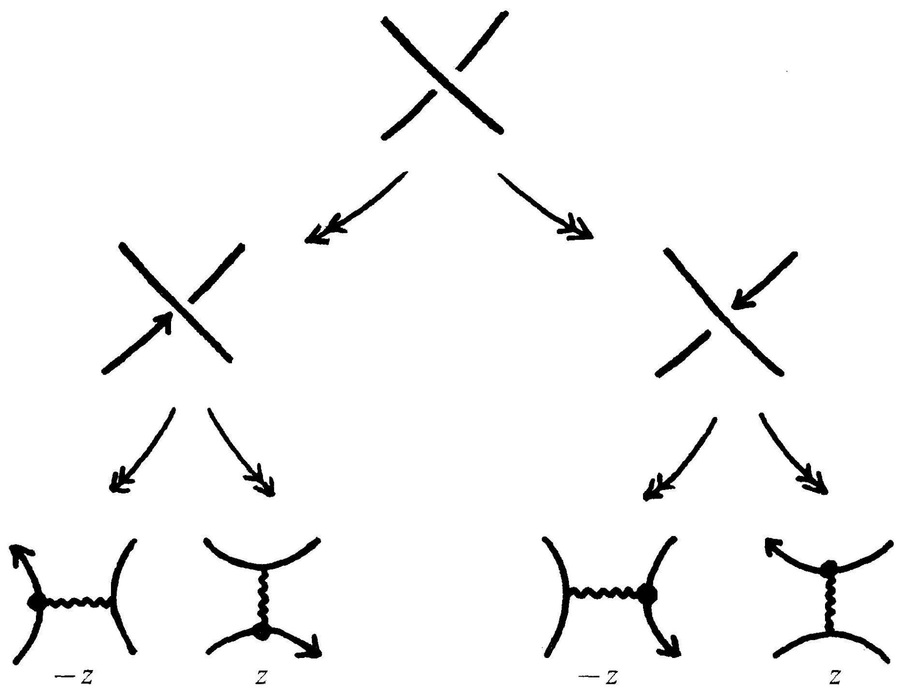


Figure 7