$\phi = \bar{\phi} + \frac{\sqrt{D}}{a} > -1$. Hence, as ϕ cannot satisfy (4.4), we must have $\phi > 1$, so I is reduced.

LEMMA 4. *If* $I = d\left[a, \frac{b+\sqrt{D}}{2}\right]$ *is an ideal of* O_D *with* $0 < a < \frac{\sqrt{D}}{2}$ *then* I *is reduced.*

Proof. We can write $I = da[1, \phi]$ with $-1 < \bar{\phi} < 0$. Then we have $\phi = \bar{\phi} + \frac{\sqrt{D}}{a} > 1$ so that I is reduced.

5. LAGRANGE'S REDUCTION PROCEDURE

In this section we describe Lagrange's reduction procedure which was first introduced in [2]. This procedure uses Lagrange neighbours and so is based on the continued fraction algorithm. The procedure, when applied to a given primitive ideal I of O_D, gives all the reduced ideals of O_D which are equivalent to I.

Let $\{a, b\}$ be a representation of the primitive ideal I of O_D. The Lagrange neighbour of $\{a, b\}$ is the representation $\{a', b'\}$ of the primitive ideal I' of O_D given as follows:

$$(5.1)\quad \begin{cases} q = [\phi] = \left[\dfrac{b+\sqrt{D}}{2a}\right], & \phi = q + \dfrac{1}{\phi'}, \\ b' = -b + 2aq, & a' = \dfrac{D - b'^2}{4a} = \dfrac{D-b^2}{4a} + bq - aq^2, \end{cases}$$

(see (2.10) and (2.11)). We write $\{a, b\} \xrightarrow{L} \{a', b'\}$. The primitive ideal $I' = a'[1, \phi']$ is also called the Lagrange neighbour of I.

We note that

$$\phi' = \frac{1}{\phi - q} > 1, [\phi'] \geqslant 1,$$

as $q = [\phi]$. We also remark that if a is kept fixed and ϕ is changed modulo 1 then ϕ', b' and a' do not change. Hence the Lagrange neighbour of $\{a, b\}$ depends only upon the sign of a. If $\{a, b\} \xrightarrow{L} \{a', b'\}$ then by Corollary 1 the ideals $I = a[1, \phi]$ and $I' = a'[1, \phi']$ are equivalent and $I' = \rho I$ with $\rho = \frac{a'}{a}\phi' = \frac{-1}{\bar{\phi}'}$.

PROPOSITION 5. *If* $\{a, b\} \xrightarrow{L} \{a', b'\}$, *where* $a > 0$ *and the ideal* $I = a[1, \phi]$ *is reduced, then the number* ϕ' *is reduced and the ideal* $I' = a'[1, \phi']$ *is reduced.*

Proof. As $a > 0$ and the ideal I is reduced, we may assume that ϕ is reduced, so that $-1 < \bar{\phi}' = \frac{1}{\bar{\phi} - q} < 0$, where $q = [\phi]$, showing that ϕ' is reduced. The ideal I' is reduced as ϕ' is reduced.

Remark. If $\{a, b\} \xrightarrow{L} \{a', b'\}$, where $a < 0$ and the ideal $I = a[1, \phi]$ is reduced, it may happen that the Lagrange neighbour $I' = a'[1, \phi']$ of I is not reduced. For example the ideal $I = [3, 7 + \sqrt{82}]$ of O_{328} is reduced and $\{-3, 14\} \xrightarrow{L} \{13, 22\}$, but the Lagrange neighbour $I' = [13, 11 + \sqrt{82}]$ of I is not reduced.

The next proposition gives information about the ideals having a specified Lagrange neighbour.

PROPOSITION 6. *(i) If* $\{a_1, b_1\} \xrightarrow{L} \{a', b'\}$ *and* $\{a_2, b_2\} \xrightarrow{L} \{a', b'\}$ *then the primitive ideals* $a_1[1, \phi_1], a_2[1, \phi_2]$ *are equal.*

(ii) If $a'[1, \phi']$ *is a primitive ideal with* $a' > 0$ *and* ϕ' *reduced, then there exists a unique reduced primitive ideal* $a[1, \phi]$ *such that* $\{a, b\} \xrightarrow{L} \{a', b'\}$.

Proof. (i) Let $q_1 = [\phi_1]$ and $q_2 = [\phi_2]$. Then we have $\phi_1 = q_1 + \frac{1}{\phi'}$ and $\phi_2 = q_2 + \frac{1}{\phi'}$, so that $\frac{b_1 + \sqrt{D}}{2a_1} = (q_1 - q_2) + \frac{b_2 + \sqrt{D}}{2a_2}$, showing that $a_1 = a_2$ and $\phi_1 \equiv \phi_2 \pmod 1$. Hence we have $a_1[1, \phi] = a_2[1, \phi_2]$.

(ii) As ϕ' is reduced we have $\phi' > 1$ and $-1 < \bar{\phi}' < 0$. Hence there is a unique integer $q (\geqslant 1)$ such that $-1 - \frac{1}{\bar{\phi}'} < q < \frac{-1}{\bar{\phi}'}$. Set $\phi = q + \frac{1}{\phi'} > 1$. It is easy to check that $\phi = \frac{b + \sqrt{D}}{2a}$, where $a, b \in Z$. Then $\bar{\phi} = q + \frac{1}{\bar{\phi}'}$ satisfies $-1 < \bar{\phi} < 0$. Thus ϕ is reduced and the ideal $a[1, \phi]$ is both primitive and reduced. Clearly $\{a, b\} \xrightarrow{L} \{a', b'\}$ and the uniqueness of the ideal $a[1, \phi]$ follows from (i).

Now that we have the notion of Lagrange neighbour and its basic properties, we can define the Lagrange reduction process, which transforms a given primitive ideal into a reduced ideal.

Definition 11. (Lagrange reduction process) We start a representation $\{a_0, b_0\}$ with $a_0 > 0$ of a primitive ideal I of O_D, and define the sequence of representations $\{a_n, b_n\}$ of the primitive ideals I_n by

$$\{a_n, b_n\} \xrightarrow{L} \{a_{n+1}, b_{n+1}\} \; (n=0, 1, 2, \ldots) \; . \tag{5.2}$$

In the Lagrange reduction process the integers q_n and the quantities ϕ_n are given by

$$q_n = [\phi_n] \; , \qquad \phi_n = \frac{b_n + \sqrt{D}}{2a_n} \, , \tag{5.3}$$

so that

$$I_n = a_n[1, \phi_n] = \left[a_n, \frac{b_n + \sqrt{D}}{2}\right] . \tag{5.4}$$

By Corollary 1, we have

$$I_n = \rho_n I_0, \rho_n = \prod_{i=1}^{n} \left(\frac{-1}{\bar{\phi}_i}\right) = \frac{a_n}{a_0} \prod_{i=1}^{n} \phi_i \, . \tag{5.5}$$

We remark that $q_n \geqslant 1$ for $n \geqslant 1$.

The next lemma tells us that if $\bar{\phi}_n$ is negative for some $n \geqslant 1$ then I_n and its successive Lagrange neighbours are all reduced.

LEMMA 5. *If* $n \geqslant 1$ *and* $\bar{\phi}_n < 0$

then

(i) $a_m > 0$, *for* $m \geqslant n - 1$,

and

(ii) $I_m = a_m[1, \phi_m]$ *is reduced for* $m \geqslant n$.

Proof. (i) As $q_n \geqslant 1$ and $\bar{\phi}_n < 0$, we see that $\bar{\phi}_{n+1} = \dfrac{1}{\bar{\phi}_n - q_n} < 0$, and so $\bar{\phi}_m < 0$ for $m \geqslant n$. For $m \geqslant n$ we have $\phi_m = \dfrac{b_m + \sqrt{D}}{2a_m} > 1$ and

$\bar{\phi}_m = \dfrac{b_m - \sqrt{D}}{2a_m} < 0$, so that $a_m > 0$ and $|b_m| < \sqrt{D}$. By (5.1) we have $D - b_m^2 = 4a_m a_{m-1} > 0$, so that $a_{m-1} > 0$. This completes the proof that $a_m > 0$ for $m \geqslant n-1$.

(ii) We have $I_m = a_m[1, \phi_m] = a_m[1, \psi_m]$, where $\psi_m = \phi_m + [|\bar{\phi}_m|]$. For $m \geqslant n \geqslant 1$, as $\psi_m \geqslant \phi_m > 1$ and $-1 < \bar{\psi}_m = \bar{\phi}_m + [|\bar{\phi}_m|] < 0$, we see that ψ_m is a reduced number, and so the ideal $I_m (m \geqslant n)$ is reduced.

Next we define two sequences of integers $\{A_n\}$ and $\{B_n\}$ for $n \geqslant -2$ by

(5.6)
$$\begin{cases} A_{-2} = 0, \quad A_{-1} = 1, \quad A_n = q_n A_{n-1} + A_{n-2}\,, \\ B_{-2} = 1, \quad B_{-1} = 0, \quad B_n = q_n B_{n-1} + B_{n-2}\,. \end{cases}$$

These sequences have the following basic properties:

(5.7)
$$\phi_n = -\left(\frac{B_{n-2}\phi_0 - A_{n-2}}{B_{n-1}\phi_0 - A_{n-1}}\right), \quad n \geqslant 0\,,$$

(5.8)
$$\phi_0 = \frac{A_{n-1}\phi_n + A_{n-2}}{B_{n-1}\phi_n + B_{n-2}}, \quad n \geqslant 0\,,$$

(5.9)
$$A_n B_{n-1} - A_{n-1} B_n = (-1)^{n-1}, \quad n \geqslant -1\,,$$

(5.10)
$$\begin{cases} B_n \geqslant \left(\dfrac{1+\sqrt{5}}{2}\right)^{n-1}, \quad n \geqslant 0\,, \\ \text{if} \quad q_0 \geqslant 1 \text{ then } A_n \geqslant \left(\dfrac{1+\sqrt{5}}{2}\right)^{n}, \quad n \geqslant 0\,, \end{cases}$$

(5.11)
$$\frac{A_n}{B_n} - \phi_0 = \frac{(-1)^{n-1}}{B_n^2 \phi_{n+1} + B_n B_{n-1}}, \quad n \geqslant 0\,,$$

(5.12)
$$(-1)^n(\phi_0 - \bar{\phi}_0) = \frac{1}{(B_{n-1}^2 \bar{\phi}_n + B_{n-1}B_{n-2})} - \frac{1}{(B_{n-1}^2 \phi_n + B_{n-1}B_{n-2})}, \quad n \geqslant 0\,,$$

(5.13)
$$\phi_1 \ldots \phi_n = B_{n-1}\phi_n + B_{n-2}\,, \quad n \geqslant 1\,.$$

We now briefly mention how these properties can be proved. The equalities (5.8) and (5.13) follow by induction using $\phi_n = q_n + \dfrac{1}{\phi_{n+1}}$. The assertion

(5.7) is just a reformulation of (5.8). The assertions (5.9) and (5.10) follow by induction using (5.6); (5.11) follows from (5.8) and (5.9); and (5.12) follows from (5.11).

The next result shows that $\bar{\phi}_n$ does eventually become negative.

LEMMA 6. (Compare [12]: Corollary 4.2.1) *Let*

$$M_0 = \max\left(\frac{1}{2}\,\frac{\operatorname{Log}(a_0/\sqrt{D})}{\operatorname{Log}((1+\sqrt{5})/2)} + \frac{5}{2}\,,\, 2\right). \tag{5.14}$$

For $n \geqslant M_0$ *we have* $\bar{\phi}_n < 0$.

Proof. For $n \geqslant M_0$, we have $n \geqslant 2$, and, appealing to (5.10) and (5.14), we obtain

$$B_{n-1}B_{n-2} \geqslant \left(\frac{1+\sqrt{5}}{2}\right)^{2n-5} \geqslant \frac{a_0}{\sqrt{D}} = \frac{1}{|\phi_0 - \bar{\phi}_0|}\,. \tag{5.15}$$

If $\bar{\phi}_n > 0$, then, by (5.12), we have

$$|\phi_0 - \bar{\phi}_0| < \max\left(\frac{1}{B_{n-1}^2\bar{\phi}_n + B_{n-1}B_{n-2}}\,,\, \frac{1}{B_{n-1}^2\phi_n + B_{n-1}B_{n-2}}\right)$$
$$< \frac{1}{B_{n-1}B_{n-2}}\,,$$

which contradicts (5.15). Hence we must have $\bar{\phi}_n < 0$, for $n \geqslant M_0$.

The next proposition gives an upper bound for the number of steps needed in the Lagrange reduction process to obtain a reduced ideal I from a given primitive ideal I_0 of O_D and at the same time gives upper and lower bounds for δ in the relation $I = \delta I_0$.

PROPOSITION 7. (Compare [12]: Theorem 4.3) *Let* $I_0 = a_0[1, \phi_0]$ *be a primitive ideal of* O_D *with* $a_0 > 0$. *Then the Lagrange reduction process applied to* I_0 *yields a reduced, primitive ideal* I *equivalent to* I_0 *with*

$$I = \delta I_0\,,\quad \frac{1}{a_0} \leqslant \delta < 2\,, \tag{5.16}$$

in atmost M_0 *steps. All the subsequent Lagrange neighbours of* I *are also reduced.*

Proof. Let n_0 be the least positive integer such that $\bar{\phi}_{n_0} < 0$. By Proposition 7 we have $n_0 \leqslant M_0$. By Lemma 5 the ideal I_{n_0} is reduced, and $a_{n_0-1} > 0$, $a_{n_0} > 0$.

We set

$$\delta = \begin{cases} \dfrac{a_{n_0-1}}{a_0} \phi_1 \dots \phi_{n_0-1}, & \text{if } I_{n_0-1} \text{ is reduced ,} \\ \dfrac{a_{n_0}}{a_0} \phi_1 \dots \phi_{n_0}, & \text{if } I_{n_0-1} \text{ is not reduced ,} \end{cases} \tag{5.17}$$

so that by (5.3) $I = \delta I_0$ is reduced, and it remains to show that $\dfrac{1}{a_0} \leqslant \delta < 2$.

For $n_0 \geqslant 2$, by (5.13), we have

$$\phi_1 \dots \phi_{n_0-1} = B_{n_0-2}\phi_{n_0-1} + B_{n_0-3} , \tag{5.18}$$

so that

$$\bar{\phi}_1 \dots \bar{\phi}_{n_0-1} = B_{n_0-2}\bar{\phi}_{n_0-1} + B_{n_0-3} > B_{n_0-3}, \tag{5.19}$$

by the definition of n_0. As $\phi_n \bar{\phi}_n = \dfrac{-a_{n-1}}{a_n}$, for $n \geqslant 1$, we have

$$(\phi_1 \dots \phi_{n_0-1})(\bar{\phi}_1 \dots \bar{\phi}_{n_0-1}) = (-1)^{n_0-1} \frac{a_0}{a_{n_0-1}} , \tag{5.20}$$

which shows (as $a_0 > 0$, $a_{n_0-1} > 0$, $\phi_i > 1\,(i \geqslant 1)$, $\bar{\phi}_i > 0(1 \leqslant i \leqslant n_0 - 1)$) that n_0 is odd. Hence $n_0 \geqslant 3$ and we have $B_{n_0-3} \geqslant 1$. Then, from (5.19) and (5.20), we obtain

$$1 < \phi_1 \dots \phi_{n_0-1} < \frac{a_0}{a_{n_0-1}} \frac{1}{B_{n_0-3}} . \tag{5.21}$$

If I_{n_0-1} is reduced then, by (5.17) and (5.21), we obtain

$$\frac{a_{n_0-1}}{a_0} < \delta < \frac{1}{B_{n_0-3}} .$$

If I_{n_0-1} is not reduced then, as $a_{n_0-1} > 0$, by Lemma 4 we have $a_{n_0-1} > \dfrac{\sqrt{D}}{2}$.

Further, as $a_{n_0} > 0$ and $D = b_{n_0}^2 + 4a_{n_0-1}a_{n_0}$, we see that $1 < \phi_{n_0} < \dfrac{\sqrt{D}}{a_{n_0}}$

$< \dfrac{2a_{n_0-1}}{a_{n_0}}$. Then, appealing to (5.20), we obtain

$$1 < \phi_1 \dots \phi_{n_0} < \frac{2a_0}{a_{n_0} B_{n_0-3}},$$

so that, by (5.17), we have

$$\frac{a_{n_0}}{a_0} < \delta < \frac{2}{B_{n_0-3}}.$$

It remains to consider the case $n_0 = 1$. If I_0 is reduced then $\delta = 1$. If I_0 is not reduced then $\delta = \dfrac{a_1}{a_0} \phi_1$ and, as above, we have $1 < \phi_1 < \dfrac{2a_0}{a_1}$, giving $\dfrac{a_1}{a_0} < \delta < 2$.

Hence in all cases we have $\dfrac{1}{a_0} \leqslant \delta < 2$. All subsequent Lagrange neighbours of I are reduced by Lemma 5. This completes the proof of Proposition 7.

6. Periods of reduced cycles

We show that any two equivalent reduced, primitive ideals of the same order O_D can be obtained from one another by using the Lagrange reduction process described in §5.

PROPOSITION 8. ([5]: §31, [12]: Theorem 4.5) *Let* $I = a[1, \phi]$ $(a > 0)$ *and* $J = b[1, \psi]$ $(b > 0)$ *be two equivalent, reduced, primitive ideals of* O_D, *so that* $[1, \psi] = \rho[1, \phi]$ *for some* $\rho(>0) \in K^*$. *Interchanging* I *and* J *if necessary we may suppose that* $\rho \geqslant 1$. *Set* $I_0 = I$. *Then there exists a non negative integer* n *such that* $J = I_n$ *and* $\rho = \phi_1 \dots \phi_n$, *so that* $J = I_n = \rho_n I$.

Proof. Recalling that $\phi_n > 1\,(n \geqslant 1)$, we see from (5.10) and (5.13) that the sequence $\{\phi_1 \dots \phi_n\}_{n=0}^{\infty}$ is monotonically increasing and unbounded. Hence there exists an integer $n \geqslant 0$ such that $\phi_1 \dots \phi_n \leqslant \rho < \phi_1 \dots \phi_{n+1}$. As $I_n = \dfrac{a_n}{a_0} \phi_1 \dots \phi_n I_0$ (by (5.5)), we have $\dfrac{1}{b} J = \dfrac{\rho}{\phi_1 \dots \phi_n} \dfrac{1}{a_n} I_n$. If $\rho = \phi_1 \dots \phi_n$ then