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$\phi = \bar{\phi} + \frac{\sqrt{D}}{a} > -1$ . Hence, as  $\phi$  cannot satisfy (4.4), we must have  $\phi > 1$ , so  $I$  is reduced.

LEMMA 4. If  $I = d \left[ a, \frac{b + \sqrt{D}}{2} \right]$  is an ideal of  $O_D$  with  $0 < a < \frac{\sqrt{D}}{2}$  then  $I$  is reduced.

*Proof.* We can write  $I = da[1, \phi]$  with  $-1 < \bar{\phi} < 0$ . Then we have  $\phi = \bar{\phi} + \frac{\sqrt{D}}{a} > 1$  so that  $I$  is reduced.

## 5. LAGRANGE'S REDUCTION PROCEDURE

In this section we describe Lagrange's reduction procedure which was first introduced in [2]. This procedure uses Lagrange neighbours and so is based on the continued fraction algorithm. The procedure, when applied to a given primitive ideal  $I$  of  $O_D$ , gives all the reduced ideals of  $O_D$  which are equivalent to  $I$ .

Let  $\{a, b\}$  be a representation of the primitive ideal  $I$  of  $O_D$ . The Lagrange neighbour of  $\{a, b\}$  is the representation  $\{a', b'\}$  of the primitive ideal  $I'$  of  $O_D$  given as follows:

$$(5.1) \quad \begin{cases} q = [\phi] = \left[ \frac{b + \sqrt{D}}{2a} \right], & \phi = q + \frac{1}{\phi'}, \\ b' = -b + 2aq, & a' = \frac{D - b'^2}{4a} = \frac{D - b^2}{4a} + bq - aq^2, \end{cases}$$

(see (2.10) and (2.11)). We write  $\{a, b\} \xrightarrow{L} \{a', b'\}$ . The primitive ideal  $I' = a'[1, \phi']$  is also called the Lagrange neighbour of  $I$ .

We note that

$$\phi' = \frac{1}{\phi - q} > 1, [\phi'] \geq 1,$$

as  $q = [\phi]$ . We also remark that if  $a$  is kept fixed and  $\phi$  is changed modulo 1 then  $\phi', b'$  and  $a'$  do not change. Hence the Lagrange neighbour of  $\{a, b\}$  depends only upon the sign of  $a$ . If  $\{a, b\} \xrightarrow{L} \{a', b'\}$  then by Corollary 1 the

ideals  $I = a[1, \phi]$  and  $I' = a'[1, \phi']$  are equivalent and  $I' = \rho I$  with  $\rho = \frac{a'}{a} \phi' = \frac{-1}{\bar{\phi}'}$ .

PROPOSITION 5. If  $\{a, b\} \xrightarrow{L} \{a', b'\}$ , where  $a > 0$  and the ideal  $I = a[1, \phi]$  is reduced, then the number  $\phi'$  is reduced and the ideal  $I' = a'[1, \phi']$  is reduced.

*Proof.* As  $a > 0$  and the ideal  $I$  is reduced, we may assume that  $\phi$  is reduced, so that  $-1 < \bar{\phi}' = \frac{1}{\bar{\phi} - q} < 0$ , where  $q = [\phi]$ , showing that  $\phi'$  is reduced. The ideal  $I'$  is reduced as  $\phi'$  is reduced.

*Remark.* If  $\{a, b\} \xrightarrow{L} \{a', b'\}$ , where  $a < 0$  and the ideal  $I = a[1, \phi]$  is reduced, it may happen that the Lagrange neighbour  $I' = a'[1, \phi']$  of  $I$  is not reduced. For example the ideal  $I = [3, 7 + \sqrt{82}]$  of  $O_{328}$  is reduced and  $\{-3, 14\} \xrightarrow{L} \{13, 22\}$ , but the Lagrange neighbour  $I' = [13, 11 + \sqrt{82}]$  of  $I$  is not reduced.

The next proposition gives information about the ideals having a specified Lagrange neighbour.

PROPOSITION 6. (i) If  $\{a_1, b_1\} \xrightarrow{L} \{a', b'\}$  and  $\{a_2, b_2\} \xrightarrow{L} \{a', b'\}$  then the primitive ideals  $a_1[1, \phi_1], a_2[1, \phi_2]$  are equal.

(ii) If  $a'[1, \phi']$  is a primitive ideal with  $a' > 0$  and  $\phi'$  reduced, then there exists a unique reduced primitive ideal  $a[1, \phi]$  such that  $\{a, b\} \xrightarrow{L} \{a', b'\}$ .

*Proof.* (i) Let  $q_1 = [\phi_1]$  and  $q_2 = [\phi_2]$ . Then we have  $\phi_1 = q_1 + \frac{1}{\phi'}$  and  $\phi_2 = q_2 + \frac{1}{\phi'}$ , so that  $\frac{b_1 + \sqrt{D}}{2a_1} = (q_1 - q_2) + \frac{b_2 + \sqrt{D}}{2a_2}$ , showing that  $a_1 = a_2$  and  $\phi_1 \equiv \phi_2 \pmod{1}$ . Hence we have  $a_1[1, \phi] = a_2[1, \phi_2]$ .

(ii) As  $\phi'$  is reduced we have  $\phi' > 1$  and  $-1 < \bar{\phi}' < 0$ . Hence there is a unique integer  $q (\geq 1)$  such that  $-1 - \frac{1}{\bar{\phi}'} < q < \frac{-1}{\bar{\phi}'}$ . Set  $\phi = q + \frac{1}{\phi'} > 1$ . It is easy to check that  $\phi = \frac{b + \sqrt{D}}{2a}$ , where  $a, b \in \mathbb{Z}$ . Then  $\bar{\phi} = q + \frac{1}{\bar{\phi}'}$  satisfies  $-1 < \bar{\phi} < 0$ . Thus  $\phi$  is reduced and the ideal  $a[1, \phi]$  is both primitive and

reduced. Clearly  $\{a, b\} \xrightarrow{L} \{a', b'\}$  and the uniqueness of the ideal  $a[1, \phi]$  follows from (i).

Now that we have the notion of Lagrange neighbour and its basic properties, we can define the Lagrange reduction process, which transforms a given primitive ideal into a reduced ideal.

*Definition 11. (Lagrange reduction process)* We start a representation  $\{a_0, b_0\}$  with  $a_0 > 0$  of a primitive ideal  $I$  of  $O_D$ , and define the sequence of representations  $\{a_n, b_n\}$  of the primitive ideals  $I_n$  by

$$(5.2) \quad \{a_n, b_n\} \xrightarrow{L} \{a_{n+1}, b_{n+1}\} \quad (n=0, 1, 2, \dots).$$

In the Lagrange reduction process the integers  $q_n$  and the quantities  $\phi_n$  are given by

$$(5.3) \quad q_n = [\phi_n], \quad \phi_n = \frac{b_n + \sqrt{D}}{2a_n},$$

so that

$$(5.4) \quad I_n = a_n[1, \phi_n] = \left[ a_n, \frac{b_n + \sqrt{D}}{2} \right].$$

By Corollary 1, we have

$$(5.5) \quad I_n = \rho_n I_0, \quad \rho_n = \prod_{i=1}^n \left( \frac{-1}{\bar{\phi}_i} \right) = \frac{a_n}{a_0} \prod_{i=1}^n \phi_i.$$

We remark that  $q_n \geq 1$  for  $n \geq 1$ .

The next lemma tells us that if  $\bar{\phi}_n$  is negative for some  $n \geq 1$  then  $I_n$  and its successive Lagrange neighbours are all reduced.

LEMMA 5. If  $n \geq 1$  and  $\bar{\phi}_n < 0$  then

(i)  $a_m > 0$ , for  $m \geq n-1$ ,

and

(ii)  $I_m = a_m[1, \phi_m]$  is reduced for  $m \geq n$ .

*Proof.* (i) As  $q_n \geq 1$  and  $\bar{\phi}_n < 0$ , we see that  $\bar{\phi}_{n+1} = \frac{1}{\bar{\phi}_n - q_n} < 0$ , and so  $\bar{\phi}_m < 0$  for  $m \geq n$ . For  $m \geq n$  we have  $\phi_m = \frac{b_m + \sqrt{D}}{2a_m} > 1$  and

$\bar{\phi}_m = \frac{b_m - \sqrt{D}}{2a_m} < 0$ , so that  $a_m > 0$  and  $|b_m| < \sqrt{D}$ . By (5.1) we have  $D - b_m^2 = 4a_m a_{m-1} > 0$ , so that  $a_{m-1} > 0$ . This completes the proof that  $a_m > 0$  for  $m \geq n-1$ .

(ii) We have  $I_m = a_m[1, \phi_m] = a_m[1, \psi_m]$ , where  $\psi_m = \phi_m + [|\bar{\phi}_m|]$ . For  $m \geq n \geq 1$ , as  $\psi_m \geq \phi_m > 1$  and  $-1 < \bar{\psi}_m = \bar{\phi}_m + [|\bar{\phi}_m|] < 0$ , we see that  $\psi_m$  is a reduced number, and so the ideal  $I_m (m \geq n)$  is reduced.

Next we define two sequences of integers  $\{A_n\}$  and  $\{B_n\}$  for  $n \geq -2$  by

$$(5.6) \quad \begin{cases} A_{-2} = 0, & A_{-1} = 1, & A_n = q_n A_{n-1} + A_{n-2}, \\ B_{-2} = 1, & B_{-1} = 0, & B_n = q_n B_{n-1} + B_{n-2}. \end{cases}$$

These sequences have the following basic properties:

$$(5.7) \quad \phi_n = - \left( \frac{B_{n-2}\phi_0 - A_{n-2}}{B_{n-1}\phi_0 - A_{n-1}} \right), \quad n \geq 0,$$

$$(5.8) \quad \phi_0 = \frac{A_{n-1}\phi_n + A_{n-2}}{B_{n-1}\phi_n + B_{n-2}}, \quad n \geq 0,$$

$$(5.9) \quad A_n B_{n-1} - A_{n-1} B_n = (-1)^{n-1}, \quad n \geq -1,$$

$$(5.10) \quad \begin{cases} B_n \geq \left( \frac{1 + \sqrt{5}}{2} \right)^{n-1}, & n \geq 0, \\ \text{if } q_0 \geq 1 \text{ then } A_n \geq \left( \frac{1 + \sqrt{5}}{2} \right)^n, & n \geq 0, \end{cases}$$

$$(5.11) \quad \frac{A_n}{B_n} - \phi_0 = \frac{(-1)^{n-1}}{B_n^2 \phi_{n+1} + B_n B_{n-1}}, \quad n \geq 0,$$

$$(5.12) \quad \begin{aligned} (-1)^n (\phi_0 - \bar{\phi}_0) &= \frac{1}{(B_{n-1}^2 \bar{\phi}_n + B_{n-1} B_{n-2})} \\ &\quad - \frac{1}{(B_{n-1}^2 \phi_n + B_{n-1} B_{n-2})}, \quad n \geq 0, \end{aligned}$$

$$(5.13) \quad \phi_1 \dots \phi_n = B_{n-1} \phi_n + B_{n-2}, \quad n \geq 1.$$

We now briefly mention how these properties can be proved. The equalities (5.8) and (5.13) follow by induction using  $\phi_n = q_n + \frac{1}{\phi_{n+1}}$ . The assertion

(5.7) is just a reformulation of (5.8). The assertions (5.9) and (5.10) follow by induction using (5.6); (5.11) follows from (5.8) and (5.9); and (5.12) follows from (5.11).

The next result shows that  $\bar{\phi}_n$  does eventually become negative.

LEMMA 6. (Compare [12]: Corollary 4.2.1) *Let*

$$(5.14) \quad M_0 = \max \left( \frac{1}{2} \frac{\text{Log}(a_0/\sqrt{D})}{\text{Log}((1+\sqrt{5})/2)} + \frac{5}{2}, 2 \right).$$

For  $n \geq M_0$  we have  $\bar{\phi}_n < 0$ .

*Proof.* For  $n \geq M_0$ , we have  $n \geq 2$ , and, appealing to (5.10) and (5.14), we obtain

$$(5.15) \quad B_{n-1}B_{n-2} \geq \left( \frac{1+\sqrt{5}}{2} \right)^{2n-5} \geq \frac{a_0}{\sqrt{D}} = \frac{1}{|\phi_0 - \bar{\phi}_0|}.$$

If  $\bar{\phi}_n > 0$ , then, by (5.12), we have

$$\begin{aligned} |\phi_0 - \bar{\phi}_0| &< \max \left( \frac{1}{B_{n-1}^2 \bar{\phi}_n + B_{n-1}B_{n-2}}, \frac{1}{B_{n-1}^2 \phi_n + B_{n-1}B_{n-2}} \right) \\ &< \frac{1}{B_{n-1}B_{n-2}}, \end{aligned}$$

which contradicts (5.15). Hence we must have  $\bar{\phi}_n < 0$ , for  $n \geq M_0$ .

The next proposition gives an upper bound for the number of steps needed in the Lagrange reduction process to obtain a reduced ideal  $I$  from a given primitive ideal  $I_0$  of  $O_D$  and at the same time gives upper and lower bounds for  $\delta$  in the relation  $I = \delta I_0$ .

PROPOSITION 7. (Compare [12]: Theorem 4.3) *Let  $I_0 = a_0[1, \phi_0]$  be a primitive ideal of  $O_D$  with  $a_0 > 0$ . Then the Lagrange reduction process applied to  $I_0$  yields a reduced, primitive ideal  $I$  equivalent to  $I_0$  with*

$$(5.16) \quad I = \delta I_0, \quad \frac{1}{a_0} \leq \delta < 2,$$

*in at most  $M_0$  steps. All the subsequent Lagrange neighbours of  $I$  are also reduced.*

*Proof.* Let  $n_0$  be the least positive integer such that  $\bar{\phi}_{n_0} < 0$ . By Proposition 7 we have  $n_0 \leq M_0$ . By Lemma 5 the ideal  $I_{n_0}$  is reduced, and  $a_{n_0-1} > 0, a_{n_0} > 0$ .

We set

$$(5.17) \quad \delta = \begin{cases} \frac{a_{n_0-1}}{a_0} \phi_1 \dots \phi_{n_0-1}, & \text{if } I_{n_0-1} \text{ is reduced,} \\ \frac{a_{n_0}}{a_0} \phi_1 \dots \phi_{n_0}, & \text{if } I_{n_0-1} \text{ is not reduced,} \end{cases}$$

so that by (5.3)  $I = \delta I_0$  is reduced, and it remains to show that  $\frac{1}{a_0} \leq \delta < 2$ .

For  $n_0 \geq 2$ , by (5.13), we have

$$(5.18) \quad \phi_1 \dots \phi_{n_0-1} = B_{n_0-2} \phi_{n_0-1} + B_{n_0-3},$$

so that

$$(5.19) \quad \bar{\phi}_1 \dots \bar{\phi}_{n_0-1} = B_{n_0-2} \bar{\phi}_{n_0-1} + B_{n_0-3} > B_{n_0-3},$$

by the definition of  $n_0$ . As  $\phi_n \bar{\phi}_n = \frac{-a_{n-1}}{a_n}$ , for  $n \geq 1$ , we have

$$(5.20) \quad (\phi_1 \dots \phi_{n_0-1}) (\bar{\phi}_1 \dots \bar{\phi}_{n_0-1}) = (-1)^{n_0-1} \frac{a_0}{a_{n_0-1}},$$

which shows (as  $a_0 > 0, a_{n_0-1} > 0, \phi_i > 1 (i \geq 1), \phi_i > 0 (1 \leq i \leq n_0 - 1)$ ) that  $n_0$  is odd. Hence  $n_0 \geq 3$  and we have  $B_{n_0-3} \geq 1$ . Then, from (5.19) and (5.20), we obtain

$$(5.21) \quad 1 < \phi_1 \dots \phi_{n_0-1} < \frac{a_0}{a_{n_0-1}} \frac{1}{B_{n_0-3}}.$$

If  $I_{n_0-1}$  is reduced then, by (5.17) and (5.21), we obtain

$$\frac{a_{n_0-1}}{a_0} < \delta < \frac{1}{B_{n_0-3}}.$$

If  $I_{n_0-1}$  is not reduced then, as  $a_{n_0-1} > 0$ , by Lemma 4 we have  $a_{n_0-1} > \frac{\sqrt{D}}{2}$ .

Further, as  $a_{n_0} > 0$  and  $D = b_{n_0}^2 + 4a_{n_0-1}a_{n_0}$ , we see that  $1 < \phi_{n_0} < \frac{\sqrt{D}}{a_{n_0}}$

$< \frac{2a_{n_0-1}}{a_{n_0}}$ . Then, appealing to (5.20), we obtain

$$1 < \phi_1 \dots \phi_{n_0} < \frac{2a_0}{a_{n_0}B_{n_0-3}},$$

so that, by (5.17), we have

$$\frac{a_{n_0}}{a_0} < \delta < \frac{2}{B_{n_0-3}}.$$

It remains to consider the case  $n_0 = 1$ . If  $I_0$  is reduced then  $\delta = 1$ . If  $I_0$  is not reduced then  $\delta = \frac{a_1}{a_0} \phi_1$  and, as above, we have  $1 < \phi_1 < \frac{2a_0}{a_1}$ , giving

$$\frac{a_1}{a_0} < \delta < 2.$$

Hence in all cases we have  $\frac{1}{a_0} \leq \delta < 2$ . All subsequent Lagrange neighbours of  $I$  are reduced by Lemma 5. This completes the proof of Proposition 7.

## 6. PERIODS OF REDUCED CYCLES

We show that any two equivalent reduced, primitive ideals of the same order  $O_D$  can be obtained from one another by using the Lagrange reduction process described in §5.

**PROPOSITION 8.** ([5]: §31, [12]: Theorem 4.5) *Let  $I = a[1, \phi]$  ( $a > 0$ ) and  $J = b[1, \psi]$  ( $b > 0$ ) be two equivalent, reduced, primitive ideals of  $O_D$ , so that  $[1, \psi] = \rho[1, \phi]$  for some  $\rho(> 0) \in K^*$ . Interchanging  $I$  and  $J$  if necessary we may suppose that  $\rho \geq 1$ . Set  $I_0 = I$ . Then there exists a non negative integer  $n$  such that  $J = I_n$  and  $\rho = \phi_1 \dots \phi_n$ , so that  $J = I_n = \rho_n I$ .*

*Proof.* Recalling that  $\phi_n > 1$  ( $n \geq 1$ ), we see from (5.10) and (5.13) that the sequence  $\{\phi_1 \dots \phi_n\}_{n=0}^{\infty}$  is monotonically increasing and unbounded. Hence there exists an integer  $n \geq 0$  such that  $\phi_1 \dots \phi_n \leq \rho < \phi_1 \dots \phi_{n+1}$ . As

$I_n = \frac{a_n}{a_0} \phi_1 \dots \phi_n I_0$  (by (5.5)), we have  $\frac{1}{b} J = \frac{\rho}{\phi_1 \dots \phi_n} \frac{1}{a_n} I_n$ . If  $\rho = \phi_1 \dots \phi_n$  then