Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 36 (1990)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE DISTANCE BETWEEN IDEALS IN THE ORDERS OF A REAL QUADRATIC FIELD
Autor: Kaplan, Pierre / Williams, Kenneth S.
Kapitel: 4. Reduced ideals
DOI: https://doi.org/10.5169/seals-57912

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 06.03.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Then we have $GCD(a, b, c) = 1$, where $\frac{D - b^2}{4a} = c$, and so $GCD(a, bf, cf^2) = 1$, showing that $I = \left[a, f\left(\frac{b + \sqrt{D}}{2}\right)\right]$ is a primitive ideal of $O_{D'}$. Hence C is the image of the class of I under θ.

COROLLARY 4. *If the class C of $O_{D'}$ contains the primitive ideal $I = \left[a, \frac{b + \sqrt{D'}}{2}\right]$, where $f^2 \mid a$, then $f \mid b$ and the class $\theta(C)$ contains the primitive ideal $J = \left[\frac{a}{f^2}, \frac{\frac{b}{f} + \sqrt{D}}{2}\right]$ of O_D.*

Proof. As $D' = Df^2 = b^2 + 4ac$, and $f^2 \mid a$, we see that $f \mid b$, and so $GCD(f, c) = 1$. By Corollary 2 we have $I = \left(\frac{\sqrt{D'} - b}{2a}\right)\left[c, \frac{-b + \sqrt{D'}}{2}\right]$ and so, by Theorem 1, we see that $\left[c, \frac{-\frac{b}{f} + \sqrt{D}}{2}\right] \in \theta(C)$. Finally, by Corollary 2, $J = \left[\frac{a}{f^2}, \frac{b/f + \sqrt{D}}{2}\right] = \frac{\left(\sqrt{D} + \frac{b}{f}\right)}{2c}\left[c, \frac{\frac{-b}{f} + \sqrt{D}}{2}\right]$, showing that $J \in \theta(C)$.

4. REDUCED IDEALS

From now on in this paper we suppose that $D_0 > 0$ so that we are only considering ideals in orders of a real quadratic field. An ideal I of O_D can be written in the form $I = ad[1, \phi]$, where $\phi = \frac{b + \sqrt{D}}{2a}$. By Proposition 1 (ii), if $I = a'd'[1, \phi']$ is another representation of I, then $a' = \pm a$ and $\phi' \equiv \frac{a}{a'}\phi \pmod 1$. A real number of the form $\frac{b + \sqrt{D}}{2a}$, where $c = \frac{D - b^2}{4a}$ is an integer and $GCD(a, b, c) = 1$ is called a quadratic irrationality of discriminant D.

Definition 9. (Reduced number). The quadratic irrationality $\phi = \dfrac{b + \sqrt{D}}{2a}$ of discriminant D is said to be *reduced* if

$$\phi > 1 , \quad -1 < \bar{\phi} < 0 . \tag{4.1}$$

It is easy to check that (4.1) is equivalent to each of the inequalities in (4.2)

$$\text{(i)} \quad 0 < \sqrt{D} - b < 2a < \sqrt{D} + b , \tag{4.2}$$

$$\text{(ii)} \quad 0 < \sqrt{D} - b < 2c < \sqrt{D} + b .$$

Moreover (4.2) implies

$$0 < a < \sqrt{D} , \quad 0 < b < \sqrt{D} , \quad 0 < c < \sqrt{D} . \tag{4.3}$$

Definition 10. (Reduced ideal). The ideal $I = ad[1, \phi]$ of O_D, where $\phi = \dfrac{b + \sqrt{D}}{2a}$, is said to be *reduced* if, and only if, ϕ can be chosen to be reduced.

From (4.3) we see that the number of reduced, primitive ideals of O_D is finite.

PROPOSITION 4. ([12]: Definition and Theorem 3.5). *The ideal*

$$I = d\left[a, \frac{b + \sqrt{D}}{2}\right]$$

of O_D, *where* $a > 0$ *and* $d > 0$, *is reduced if, and only if,* I *does not contain a nonzero element* α *satisfying* $|\alpha| < da, |\bar{\alpha}| < da$.

Proof. It suffices to prove that I is reduced if, and only if, the Z-module $[1, \phi]$ does not contain a nonzero element $\lambda = x + y\phi$ such that

$$|\lambda| < 1 , \quad |\bar{\lambda}| < 1 . \tag{4.4}$$

If I is reduced we can suppose that $\phi > 1$, $-1 < \bar{\phi} < 0$. Let x and y be integers such that $0 < \lambda = x + y\phi < 1$.

Clearly we have $y \neq 0$. If $y \geqslant 1$, then we have $y\phi > 1$, so $x \leqslant -1$, showing that $\bar{\lambda} = x + y\bar{\phi} < -1$. If $y \leqslant -1$, then we have $y\phi < -1$, so $x \geqslant 2$, showing that $\bar{\lambda} = x + y\bar{\phi} > 2$. This proves that $[1, \phi]$ does not contain an element $\lambda \neq 0$ such that $|\lambda| < 1, |\bar{\lambda}| < 1$.

Now suppose the Z-module $[1, \phi]$ does not contain an element $\lambda \neq 0$ satisfying (4.4). We can choose ϕ so that $-1 < \bar{\phi} < 0$, in which case

$\phi = \bar{\phi} + \dfrac{\sqrt{D}}{a} > -1$. Hence, as ϕ cannot satisfy (4.4), we must have $\phi > 1$, so I is reduced.

LEMMA 4. *If* $I = d\left[a, \dfrac{b+\sqrt{D}}{2}\right]$ *is an ideal of* O_D *with* $0 < a < \dfrac{\sqrt{D}}{2}$ *then* I *is reduced.*

Proof. We can write $I = da[1, \phi]$ with $-1 < \bar{\phi} < 0$. Then we have $\phi = \bar{\phi} + \dfrac{\sqrt{D}}{a} > 1$ so that I is reduced.

5. LAGRANGE'S REDUCTION PROCEDURE

In this section we describe Lagrange's reduction procedure which was first introduced in [2]. This procedure uses Lagrange neighbours and so is based on the continued fraction algorithm. The procedure, when applied to a given primitive ideal I of O_D, gives all the reduced ideals of O_D which are equivalent to I.

Let $\{a, b\}$ be a representation of the primitive ideal I of O_D. The Lagrange neighbour of $\{a, b\}$ is the representation $\{a', b'\}$ of the primitive ideal I' of O_D given as follows:

$$(5.1)\quad \begin{cases} q = [\phi] = \left[\dfrac{b+\sqrt{D}}{2a}\right], & \phi = q + \dfrac{1}{\phi'}, \\ b' = -b + 2aq, & a' = \dfrac{D-b'^2}{4a} = \dfrac{D-b^2}{4a} + bq - aq^2, \end{cases}$$

(see (2.10) and (2.11)). We write $\{a, b\} \xrightarrow{L} \{a', b'\}$. The primitive ideal $I' = a'[1, \phi']$ is also called the Lagrange neighbour of I.

We note that

$$\phi' = \frac{1}{\phi - q} > 1, [\phi'] \geqslant 1,$$

as $q = [\phi]$. We also remark that if a is kept fixed and ϕ is changed modulo 1 then ϕ', b' and a' do not change. Hence the Lagrange neighbour of $\{a, b\}$ depends only upon the sign of a. If $\{a, b\} \xrightarrow{L} \{a', b'\}$ then by Corollary 1 the