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QUADRATIC FIELD

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by Proposition 3, we have
$$J = \rho I$$
, where $\psi = \frac{p\phi + q}{r\phi + s}$, $\rho = \frac{A}{a} \frac{1}{r\phi + s}$
 $= \varepsilon(r\bar{\phi} + s)$ and $\varepsilon = ps - qr = \pm 1$. Clearly we have
$$A = \varepsilon a(r\phi + s) (r\bar{\phi} + s) = \varepsilon(as^2 + bsr - cr^2),$$

$$B = A(\psi + \bar{\psi}) = \varepsilon a(\psi + \bar{\psi}) (r\phi + s) (r\bar{\phi} + s)$$

$$= \varepsilon a((p\phi + q) (r\bar{\phi} + s) + (p\bar{\phi} + q) (r\phi + s))$$

$$= \varepsilon(2asq + b(sp + rq) - 2cpr),$$

$$-C = A\psi\bar{\psi} = \varepsilon a\psi\bar{\psi}(r\phi + s) (r\bar{\phi} + s) = \varepsilon a(p\phi + q) (p\bar{\phi} + q)$$

$$= \varepsilon(ag^2 + bap - cp^2).$$

Thus A, B, C are integral linear combinations of a, b, c. Similarly, a, b, c are integral linear combinations of A, B, C. Hence GCD(A, B, C) = GCD(a, b, c) = 1 so that J is primitive.

3. The homomorphism θ

Let O_D and $O_{D'}$ be two orders of O_{D_0} with $O_{D'} \subset O_D$. Then we have $D' = Df^2$ for some positive integer f. This notation will be used throughout the rest of the paper. Our aim is to define a surjective homomorphism θ from the ideal class group $C_{D'}$ onto the ideal class group C_D . After proving three lemmas, we will prove the following theorem.

THEOREM 1. (i) Every class C of $C_{D'}$ contains a primitive ideal I of the form $I = \left[a, \frac{fb + \sqrt{D'}}{2} \right]$, where GCD(a, f) = 1, such that the ideal $J = \left[a, \frac{b + \sqrt{D}}{2} \right]$ is a primitive ideal of O_D .

(ii) If
$$I = \left[a, \frac{fb + \sqrt{D'}}{2}\right]$$
 (GCD(a, f) = 1) and $I' = \left[a', \frac{fb' + \sqrt{D'}}{2}\right]$ (GCD(a', f) = 1) are two primitive ideals in the same class C of $C_{D'}$

with $I' = \rho I(\rho \in K^*)$, then the ideals

$$J = \left[a, \frac{b + \sqrt{D}}{2}\right]$$
 and $J' = \left[a', \frac{b' + \sqrt{D}}{2}\right]$

of O_D satisfy $J' = \rho J$ and are in the same class $\theta(C)$ of C_D .

(iii) The mapping $C \to \theta(C)$ is a homomorphism of $C_{D'}$ to on C_D . Part (ii) of Theorem 1 will be the main tool in relating distances between ideals of different orders of the same real quadratic field.

LEMMA 1. A primitive ideal $I = \left[a, \frac{b + \sqrt{D}}{2} \right]$ contains a number $\alpha = xa + y \left(\frac{b + \sqrt{D}}{2} \right)$, where x and y are coprime integers, such that the integer $N(\alpha)/a$ is prime to a given nonzero integer m.

Proof. We begin by noting that $\frac{1}{a}N\left(xa+y\left(\frac{b+\sqrt{D}}{2}\right)\right)=ax^2+bxy$ $-cy^2$ in view of (2.5). If |m|=1 we take $x=1,\ y=0,\ \alpha=xa+y\left(\frac{b+\sqrt{D}}{2}\right)=a$, so that $GCD(N(\alpha)/a,m)=GCD(a,1)=1$, as required. Hence we may suppose that |m|>1. Let $p_i(i=1,2,\dots,n)$ be the distinct

Hence we may suppose that |m| > 1. Let $p_i (i = 1, 2, ..., n)$ be the distinct prime factors of m. For i = 1, 2, ..., n we set

$$(x_i, y_i) = \begin{cases} (1,0), & \text{if} & p_i \nmid a, \\ (0,1), & \text{if} & p_i \mid a, p_i \nmid c, \\ (1,1), & \text{if} & p_i \mid a, p_i \mid c, \end{cases}$$

so that $p_i \not\mid ax_i^2 + bx_iy_i - cy_i^2$. Let x' and y' be integers such that $x' \equiv x_i \pmod{p_i}$ and $y' \equiv y_i \pmod{p_i}$ for i = 1, 2, ..., n, so that $GCD(ax'^2 + bx'y' - cy'^2, m) = 1$. The required number α is given by $\alpha = xa + y\left(\frac{b + \sqrt{D}}{2}\right)$, where $x = \frac{x'}{GCD(x', y')}$, $y = \frac{y'}{GCD(x', y')}$.

LEMMA 2. Let m be a given nonzero integer. Every class C of C_D contains a primitive ideal $\left[a, \frac{b+\sqrt{D}}{2}\right]$ with GCD(a, m) = 1.

Proof. Let $\left[a', \frac{b' + \sqrt{D}}{2}\right]$ be a primitive ideal of the class C. By Lemma 1 there exist coprime integers x and y such that

(3.1)
$$GCD(a'x^2 + b'xy - c'y^2, m) = 1.$$

Set $a = a'x^2 + b'xy - c'y^2$ and let r and s be integers such that xs - yr = 1. Next set

(3.2)
$$\rho = x + \left(\frac{b' - \sqrt{D}}{2a'}\right) y, b = 2a'xr + b'(xs + yr) - 2c'ys,$$

so that

$$a = \rho \left(xa' + y \left(\frac{b' + \sqrt{D}}{2} \right) \right)$$

and

$$\frac{b+\sqrt{D}}{2}=\rho\left(ra'+s\left(\frac{b'+\sqrt{D}}{2}\right)\right)\;.$$

Then we have

$$\left[a, \frac{b+\sqrt{D}}{2}\right] = \rho \left[xa' + y\left(\frac{b'+\sqrt{D}}{2}\right), ra' + s\left(\frac{b'+\sqrt{D}}{2}\right)\right]$$
$$= \rho \left[a', \frac{b'+\sqrt{D}}{2}\right]$$

so that $\left[a, \frac{b+\sqrt{D}}{2}\right]$ is an ideal equivalent to the primitive ideal $\left[a', \frac{b'+\sqrt{D}}{2}\right]$. Hence, by Corollary 3, $\left[a, \frac{b+\sqrt{D}}{2}\right]$ is primitive.

Lemma 3. Let C and C' be two classes of C_D . Then there exist primitive ideals $I = \left[a, \frac{B + \sqrt{D}}{2} \right] \in C$ and $I' = \left[a', \frac{B + \sqrt{D}}{2} \right] \in C'$ with GCD(a, a') = 1. Moreover the ideal II' is primitive and $II' = \left[aa', \frac{B + \sqrt{D}}{2} \right]$.

Proof. By Lemma 2 there exist primitive ideals $I = \left[a, \frac{b + \sqrt{D}}{2}\right] \in C$ and $I' = \left[a', \frac{b' + \sqrt{D}}{2}\right] \in C'$ with GCD(a, a') = 1. As $b \equiv D \equiv b' \pmod{2}$ and GCD(a, a') = 1 there are integers k and k' such that $k'a' - ka = \frac{b - b'}{2}$. Set B = b + 2ka = b' + 2k'a' so that

$$I = \left[a, \frac{B + \sqrt{D}}{2}\right]$$
 and $I' = \left[a', \frac{B + \sqrt{D}}{2}\right]$.

Now $D-B^2$ is divisible by both 4a and 4a', and so, as GCD(a, a')=1, $D-B^2$ is a multiple of 4aa', so that $c''=\frac{D-B^2}{4aa'}\in Z$. Hence $\left[aa',\frac{B+\sqrt{D}}{2}\right]$ is an ideal of O_D and we have

$$II' = \left(aa', a\left(\frac{B+\sqrt{D}}{2}\right), a'\left(\frac{B+\sqrt{D}}{2}\right), \left(\frac{B+\sqrt{D}}{2}\right)^{2}\right)$$

$$= \left(aa', \frac{B+\sqrt{D}}{2}\right)$$

$$= \left[aa', \frac{B+\sqrt{D}}{2}\right].$$

Finally, any prime divisor of aa', B, c'' must divide GCD(a, B, a'c'') = 1 or GCD(a', B, ac'') = 1, as GCD(a, a') = 1, which is impossible. Hence the ideal II' is primitive.

We are now ready to prove Theorem 1.

Proof of Theorem 1. (i) By Lemma 2 the class C contains a primitive ideal $I = \left[a, \frac{b' + \sqrt{D'}}{2} \right]$ with GCD(a, f) = 1. Let k be an integer such that

$$\begin{cases} 2ak \equiv -b' \pmod{f}, & \text{if} \quad f \equiv 1 \pmod{2}, \\ ak \equiv -\frac{b'}{2} + D \frac{f}{2} \pmod{f}, & \text{if} \quad f \equiv 0 \pmod{2}, \end{cases}$$

and set b = (b' + 2ak)/f, so that $I = \left[a, \frac{fb + \sqrt{D'}}{2}\right]$. As I is an ideal of $O_{D'}$, $(D' - f^2b^2)/4a$ is an integer, and so, as GCD(a, f) = 1, $c = (D - b^2)/4a$ is also an integer, showing that $J = \left[a, \frac{b + \sqrt{D}}{2}\right]$ is an ideal of O_D . Further, as I is primitive, we have $GCD(a, bf, cf^2) = 1$, and so GCD(a, b, c) = 1, showing that J is primitive.

(ii) If $I' = \rho I$, by Proposition 3, there exist integers p, q, r, s with $ps - qr = \pm 1$ such that

$$\frac{fb'+\sqrt{D'}}{2a'} = \frac{p\left(\frac{fb+\sqrt{D'}}{2a}\right)+q}{r\left(\frac{fb+\sqrt{D'}}{2a}\right)+s}, \quad \rho = \pm \left(r\left(\frac{fb-\sqrt{D'}}{2a}\right)+s\right).$$

Rearranging the first equation in (3.3), we obtain the following equality among elements of \mathcal{O}_D

$$f\left(\frac{b'+\sqrt{D}}{2}\right)\left(rf\left(\frac{b+\sqrt{D}}{2}\right)+sa\right)=a'\left(pf\left(\frac{b+\sqrt{D}}{2}\right)+qa\right)\,,$$

from which we deduce that $f \mid qaa'$. As GCD(aa', f) = 1 there exists an integer q' such that q = q'f, so (3.3) can be rewritten as

$$\frac{b'+\sqrt{D}}{2a'} = \frac{p\left(\frac{b+\sqrt{D}}{2a}\right)+q'}{rf\left(\frac{b+\sqrt{D}}{2a}\right)+s} , \quad \rho = \pm \left(rf\left(\frac{b-\sqrt{D}}{2a}\right)+s\right).$$

which, by Proposition 3, shows that $J' = \rho J$.

(iii) Let $C \in C_{D'}$ and $C' \in C_{D'}$. By Lemma 2 and (i), we can choose an ideal $I = \left[a, f\left(\frac{b + \sqrt{D}}{2}\right)\right]$ in C with GCD(a, f) = 1 and then an ideal $I' = \left[a', f\left(\frac{b' + \sqrt{D}}{2}\right)\right]$ in C' with GCD(a', af) = 1. By (i) $\left[a, \frac{b + \sqrt{D}}{2}\right]$ and $\left[a', \frac{b' + \sqrt{D}}{2}\right]$ are ideals of O_D and so we have $b \equiv b' \pmod{2}$. We choose integers K' and K such that $K'a' - Ka = \frac{b - b'}{2}$, and set B = b + 2Ka = b' + 2K'a', so that $I = \left[a, f\left(\frac{B + \sqrt{D}}{2}\right)\right]$ and $I' = \left[a, f\left(\frac{B + \sqrt{D}}{2}\right)\right]$. By Lemma 3 we see that $II' = \left[aa', f\left(\frac{B + \sqrt{D}}{2}\right)\right]$ is a primitive ideal of the class CC'. But the primitive ideals $J = \left[a, \frac{B + \sqrt{D}}{2}\right]$, $J' = \left[a', \frac{B + \sqrt{D}}{2}\right]$, $J'' = \left[aa', \frac{B + \sqrt{D}}{2}\right]$ belong respectively to the classes $\theta(C)$, $\theta(C')$, $\theta(CC')$, and, as JJ' = J'' by Lemma 3, we have $\theta(C)$ $\theta(C') = \theta(CC')$, showing that θ is a homomorphism: $C_{D'} \to C_D$.

Finally we show that θ is surjective. Let C be a class of C_D and let $J = \left[a, \frac{b + \sqrt{D}}{2} \right]$ be a primitive ideal of C with GCD(a, f) = 1 (Lemma 2).

Then we have GCD(a, b, c) = 1, where $\frac{D - b^2}{4a} = c$, and so $GCD(a, bf, cf^2) = 1$, showing that $I = \left[a, f\left(\frac{b + \sqrt{D}}{2}\right)\right]$ is a primitive ideal of $O_{D'}$. Hence C is the image of the class of I under θ .

COROLLARY 4. If the class C of $O_{D'}$ contains the primitive ideal $I = \left[a, \frac{b + \sqrt{D'}}{2} \right]$, where $f^2 \mid a$, then $f \mid b$ and the class $\theta(C)$ contains the primitive ideal $J = \left[\frac{a}{f^2}, \frac{b}{f} + \sqrt{D} \right]$ of O_D .

Proof. As $D' = Df^2 = b^2 + 4ac$, and $f^2 \mid a$, we see that $f \mid b$, and so GCD(f,c) = 1. By Corollary 2 we have $I = \left(\frac{\sqrt{D'} - b}{2a}\right) \left[c, \frac{-b + \sqrt{D'}}{2}\right]$ and so, by Theorem 1, we see that $\left[c, \frac{-\frac{b}{f} + \sqrt{D}}{2}\right] \in \theta(C)$. Finally, by Corollary 2, $J = \left[\frac{a}{f^2}, \frac{b/f + \sqrt{D}}{2}\right] = \frac{\left(\sqrt{D} + \frac{b}{f}\right)}{2c} \left[c, \frac{-\frac{b}{f} + \sqrt{D}}{2}\right]$, showing that $J \in \theta(C)$.

4. REDUCED IDEALS

From now on in this paper we suppose that $D_0 > 0$ so that we are only considering ideals in orders of a real quadratic field. An ideal I of O_D can be written in the form $I = ad[1, \phi]$, where $\phi = \frac{b + \sqrt{D}}{2a}$. By Proposition 1 (ii), if $I = a'd'[1, \phi']$ is another representation of I, then $a' = \pm a$ and $\phi' \equiv \frac{a}{a'} \phi$ (mod 1). A real number of the form $\frac{b + \sqrt{D}}{2a}$, where $c = \frac{D - b^2}{4a}$ is an integer and GCD(a, b, c) = 1 is called a quadratic irrationality of discriminant D.