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by Proposition 3, we have  $J = \rho I$ , where  $\psi = \frac{p\varphi + q}{r\varphi + s}$ ,  $\rho = \frac{A}{a} \frac{1}{r\varphi + s}$   $= \varepsilon(r\overline{\varphi} + s)$  and  $\varepsilon = ps - qr = \pm 1$ . Clearly we have  $A = \varepsilon a(r\varphi + s) (r\overline{\varphi} + s) = \varepsilon(as^2 + bsr - cr^2)$ ,  $B = A(\psi + \overline{\psi}) = \varepsilon a(\psi + \overline{\psi}) (r\varphi + s) (r\overline{\varphi} + s)$   $= \varepsilon a((p\varphi + q) (r\overline{\varphi} + s) + (p\overline{\varphi} + q) (r\varphi + s))$   $= \varepsilon(2asq + b(sp + rq) - 2cpr)$ ,  $-C = A\psi\overline{\psi} = \varepsilon a\psi\overline{\psi}(r\varphi + s) (r\overline{\varphi} + s) = \varepsilon a(p\varphi + q) (p\overline{\varphi} + q)$  $= \varepsilon(aq^2 + bqp - cp^2)$ .

Thus A, B, C are integral linear combinations of a, b, c. Similarly, a, b, c are integral linear combinations of A, B, C. Hence GCD(A, B, C) = GCD(a, b, c) = 1 so that J is primitive.

## 3. The homomorphism $\theta$

Let  $O_D$  and  $O_{D'}$  be two orders of  $O_{D_0}$  with  $O_{D'} \,\subset O_D$ . Then we have  $D' = Df^2$  for some positive integer f. This notation will be used throughout the rest of the paper. Our aim is to define a surjective homomorphism  $\theta$  from the ideal class group  $C_{D'}$  onto the ideal class group  $C_D$ . After proving three lemmas, we will prove the following theorem.

THEOREM 1. (i) Every class C of  $C_{D'}$  contains a primitive ideal Iof the form  $I = \left[a, \frac{fb + \sqrt{D'}}{2}\right]$ , where GCD(a, f) = 1, such that the ideal  $J = \left[a, \frac{b + \sqrt{D}}{2}\right]$  is a primitive ideal of  $O_D$ . (ii) If  $I = \left[a, \frac{fb + \sqrt{D'}}{2}\right]$  (GCD(a, f) = 1) and  $I' = \left[a', \frac{fb' + \sqrt{D'}}{2}\right]$ (GCD(a', f) = 1) are two primitive ideals in the same class C of  $C_{D'}$ with  $I' = \rho I(\rho \in K^*)$ , then the ideals

$$J = \left[a, \frac{b + \sqrt{D}}{2}\right] \quad and \quad J' = \left[a', \frac{b' + \sqrt{D}}{2}\right]$$

of  $O_D$  satisfy  $J' = \rho J$  and are in the same class  $\theta(C)$  of  $C_D$ .

(iii) The mapping  $C \rightarrow \theta(C)$  is a homomorphism of  $C_{D'}$  to on  $C_D$ . Part (ii) of Theorem 1 will be the main tool in relating distances between ideals of different orders of the same real quadratic field.

LEMMA 1. A primitive ideal  $I = \left[a, \frac{b + \sqrt{D}}{2}\right]$  contains a number  $\alpha = xa + y\left(\frac{b + \sqrt{D}}{2}\right)$ , where x and y are coprime integers, such that the integer  $N(\alpha)/a$  is prime to a given nonzero integer m.

*Proof.* We begin by noting that  $\frac{1}{a}N\left(xa + y\left(\frac{b+\sqrt{D}}{2}\right)\right) = ax^2 + bxy$ -  $cy^2$  in view of (2.5). If |m| = 1 we take x = 1, y = 0,  $\alpha = xa$ +  $y\left(\frac{b+\sqrt{D}}{2}\right) = a$ , so that  $GCD(N(\alpha)/a, m) = GCD(a, 1) = 1$ , as required.

Hence we may suppose that |m| > 1. Let  $p_i(i = 1, 2, ..., n)$  be the distinct prime factors of m. For i = 1, 2, ..., n we set

$$(x_i, y_i) = \begin{cases} (1, 0), & \text{if } p_i \not\mid a, \\ (0, 1), & \text{if } p_i \mid a, p_i \not\mid c, \\ (1, 1), & \text{if } p_i \mid a, p_i \mid c, \end{cases}$$

so that  $p_i \not\mid ax_i^2 + bx_iy_i - cy_i^2$ . Let x' and y' be integers such that  $x' \equiv x_i \pmod{p_i}$  and  $y' \equiv y_i \pmod{p_i}$  for i = 1, 2, ..., n, so that  $GCD(ax'^2 + bx'y' - cy'^2, m) = 1$ . The required number  $\alpha$  is given by  $\alpha = xa + y \left(\frac{b + \sqrt{D}}{2}\right)$ , where  $x = \frac{x'}{GCD(x', y')}$ ,  $y = \frac{y'}{GCD(x', y')}$ .

LEMMA 2. Let *m* be a given nonzero integer. Every class *C* of  $C_D$  contains a primitive ideal  $\left[a, \frac{b+\sqrt{D}}{2}\right]$  with GCD(a, m) = 1.

*Proof.* Let  $\left[a', \frac{b' + \sqrt{D}}{2}\right]$  be a primitive ideal of the class C. By Lemma 1 there exist coprime integers x and y such that

(3.1) 
$$GCD(a'x^2 + b'xy - c'y^2, m) = 1$$

Set  $a = a'x^2 + b'xy - c'y^2$  and let r and s be integers such that xs - yr = 1. Next set

(3.2) 
$$\rho = x + \left(\frac{b' - \sqrt{D}}{2a'}\right) y, b = 2a'xr + b'(xs + yr) - 2c'ys,$$

so that

$$a = \rho \left( xa' + y \left( \frac{b' + \sqrt{D}}{2} \right) \right)$$

and

$$\frac{b+\sqrt{D}}{2} = \rho \left( ra' + s \left( \frac{b'+\sqrt{D}}{2} \right) \right) .$$

Then we have

$$\begin{bmatrix} a, \frac{b+\sqrt{D}}{2} \end{bmatrix} = \rho \left[ xa' + y \left( \frac{b' + \sqrt{D}}{2} \right), ra' + s \left( \frac{b' + \sqrt{D}}{2} \right) \right]$$
$$= \rho \left[ a', \frac{b' + \sqrt{D}}{2} \right]$$

so that  $\left[a, \frac{b+\sqrt{D}}{2}\right]$  is an ideal equivalent to the primitive ideal  $\left[a', \frac{b'+\sqrt{D}}{2}\right]$ . Hence, by Corollary 3,  $\left[a, \frac{b+\sqrt{D}}{2}\right]$  is primitive.

LEMMA 3. Let C and C' be two classes of  $C_D$ . Then there exist primitive ideals  $I = \left[a, \frac{B + \sqrt{D}}{2}\right] \in C$  and  $I' = \left[a', \frac{B + \sqrt{D}}{2}\right] \in C'$ with GCD(a, a') = 1. Moreover the ideal II' is primitive and  $II' = \left[aa', \frac{B + \sqrt{D}}{2}\right]$ .

*Proof.* By Lemma 2 there exist primitive ideals  $I = \left[a, \frac{b+\sqrt{D}}{2}\right] \in C$ and  $I' = \left[a', \frac{b'+\sqrt{D}}{2}\right] \in C'$  with GCD(a, a') = 1. As  $b \equiv D \equiv b' \pmod{2}$ and GCD(a, a') = 1 there are integers k and k' such that  $k'a' - ka = \frac{b-b'}{2}$ . Set B = b + 2ka = b' + 2k'a' so that

$$I = \left[a, \frac{B + \sqrt{D}}{2}\right]$$
 and  $I' = \left[a', \frac{B + \sqrt{D}}{2}\right]$ 

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Now  $D - B^2$  is divisible by both 4a and 4a', and so, as GCD(a, a') = 1,  $D - B^2$  is a multiple of 4aa', so that  $c'' = \frac{D - B^2}{4aa'} \in \mathbb{Z}$ . Hence  $\left[aa', \frac{B + \sqrt{D}}{2}\right]$ is an ideal of  $O_D$  and we have

$$II' = \left(aa', a\left(\frac{B+\sqrt{D}}{2}\right), a'\left(\frac{B+\sqrt{D}}{2}\right), \left(\frac{B+\sqrt{D}}{2}\right)^2\right)$$
$$= \left(aa', \frac{B+\sqrt{D}}{2}\right)$$
$$= \left[aa', \frac{B+\sqrt{D}}{2}\right].$$

Finally, any prime divisor of aa', B, c'' must divide GCD(a, B, a'c'') = 1 or GCD(a', B, ac'') = 1, as GCD(a, a') = 1, which is impossible. Hence the ideal *II'* is primitive.

We are now ready to prove Theorem 1.

*Proof of Theorem 1.* (i) By Lemma 2 the class C contains a primitive ideal  $I = \left[a, \frac{b' + \sqrt{D'}}{2}\right]$  with GCD(a, f) = 1. Let k be an integer such that

$$\begin{cases} 2ak \equiv -b' \pmod{f}, & \text{if } f \equiv 1 \pmod{2}, \\ ak \equiv -\frac{b'}{2} + D \frac{f}{2} \pmod{f}, & \text{if } f \equiv 0 \pmod{2}, \end{cases}$$

and set b = (b'+2ak)/f, so that  $I = \left[a, \frac{fb+\sqrt{D'}}{2}\right]$ . As I is an ideal of  $O_{D'}, (D'-f^2b^2)/4a$  is an integer, and so, as  $GCD(a, f) = 1, c = (D-b^2)/4a$  is also an integer, showing that  $J = \left[a, \frac{b+\sqrt{D}}{2}\right]$  is an ideal of  $O_D$ . Further, as I is primitive, we have  $GCD(a, bf, cf^2) = 1$ , and so GCD(a, b, c) = 1, showing that J is primitive.

(ii) If  $I' = \rho I$ , by Proposition 3, there exist integers p, q, r, s with  $ps - qr = \pm 1$  such that

$$\frac{fb'+\sqrt{D'}}{2a'} = \frac{p\left(\frac{fb+\sqrt{D'}}{2a}\right)+q}{r\left(\frac{fb+\sqrt{D'}}{2a}\right)+s}, \quad \rho = \pm \left(r\left(\frac{fb-\sqrt{D'}}{2a}\right)+s\right).$$

Rearranging the first equation in (3.3), we obtain the following equality among elements of  $O_D$ 

$$f\left(\frac{b'+\sqrt{D}}{2}\right)\left(rf\left(\frac{b+\sqrt{D}}{2}\right)+sa\right) = a'\left(pf\left(\frac{b+\sqrt{D}}{2}\right)+qa\right),$$

from which we deduce that f | qaa'. As GCD(aa', f) = 1 there exists an integer q' such that q = q'f, so (3.3) can be rewritten as

$$\frac{b'+\sqrt{D}}{2a'} = \frac{p\left(\frac{b+\sqrt{D}}{2a}\right)+q'}{rf\left(\frac{b+\sqrt{D}}{2a}\right)+s} , \quad \rho = \pm \left(rf\left(\frac{b-\sqrt{D}}{2a}\right)+s\right) .$$

which, by Proposition 3, shows that  $J' = \rho J$ .

(iii) Let  $C \in C_{D'}$  and  $C' \in C_{D'}$ . By Lemma 2 and (i), we can choose an ideal  $I = \left[a, f\left(\frac{b+\sqrt{D}}{2}\right)\right]$  in C with GCD(a, f) = 1 and then an ideal  $I' = \left[a', f\left(\frac{b'+\sqrt{D}}{2}\right)\right]$  in C' with GCD(a', af) = 1. By (i)  $\left[a, \frac{b+\sqrt{D}}{2}\right]$  and  $\left[a', \frac{b'+\sqrt{D}}{2}\right]$  are ideals of  $O_D$  and so we have  $b \equiv b' \pmod{2}$ . We choose integers K' and K such that  $K'a' - Ka = \frac{b-b'}{2}$ , and set B = b + 2Ka = b' + 2K'a', so that  $I = \left[a, f\left(\frac{B+\sqrt{D}}{2}\right)\right]$  and  $I' = \left[a, f\left(\frac{B+\sqrt{D}}{2}\right)\right]$ . By Lemma 3 we see that  $II' = \left[aa', f\left(\frac{B+\sqrt{D}}{2}\right)\right]$  is a primitive ideal of the class CC'. But the primitive ideals  $J = \left[a, \frac{B+\sqrt{D}}{2}\right]$ ,  $J' = \left[a', \frac{B+\sqrt{D}}{2}\right]$ ,  $J'' = \left[aa', \frac{B+\sqrt{D}}{2}\right]$  belong respectively to the classes  $\theta(C)$ ,  $\theta(C')$ ,  $\theta(CC')$ , and, as JJ' = J'' by Lemma 3, we have  $\theta(C) \theta(C') = \theta(CC')$ , showing that  $\theta$  is a homomorphism:  $C_{D'} \to C_D$ .

Finally we show that  $\theta$  is surjective. Let C be a class of  $C_D$  and let  $J = \left[a, \frac{b + \sqrt{D}}{2}\right]$  be a primitive ideal of C with GCD(a, f) = 1 (Lemma 2).

Then we have GCD(a, b, c) = 1, where  $\frac{D - b^2}{4a} = c$ , and so  $GCD(a, bf, cf^2) = 1$ , showing that  $I = \left[a, f\left(\frac{b + \sqrt{D}}{2}\right)\right]$  is a primitive ideal of  $O_{D'}$ . Hence C is the image of the class of I under  $\theta$ .

COROLLARY 4. If the class C of  $O_{D'}$  contains the primitive ideal  $I = \left[a, \frac{b+\sqrt{D'}}{2}\right]$ , where  $f^2 \mid a$ , then  $f \mid b$  and the class  $\theta(C)$ contains the primitive ideal  $J = \left[\frac{a}{f^2}, \frac{b}{f} + \sqrt{D}\right]$  of  $O_D$ . Proof. As  $D' = Df^2 = b^2 + 4ac$ , and  $f^2 \mid a$ , we see that  $f \mid b$ , and so

*GCD*(*f*, *c*) = 1. By Corollary 2 we have  $I = \left(\frac{\sqrt{D'} - b}{2a}\right) \left[c, \frac{-b + \sqrt{D'}}{2}\right]$ and so, by Theorem 1, we see that  $\left[c, \frac{-\frac{b}{f} + \sqrt{D}}{2}\right] \in \Theta(C)$ . Finally, by Corollary 2,  $J = \left[\frac{a}{f^2}, \frac{b/f + \sqrt{D}}{2}\right] = \frac{\left(\sqrt{D} + \frac{b}{f}\right)}{2c} \left[c, \frac{-\frac{b}{f} + \sqrt{D}}{2}\right]$ , showing that  $J \in \Theta(C)$ .

# 4. REDUCED IDEALS

From now on in this paper we suppose that  $D_0 > 0$  so that we are only considering ideals in orders of a real quadratic field. An ideal I of  $O_D$  can be written in the form  $I = ad[1, \phi]$ , where  $\phi = \frac{b + \sqrt{D}}{2a}$ . By Proposition 1 (ii), if  $I = a'd'[1, \phi']$  is another representation of I, then  $a' = \pm a$  and  $\phi' \equiv \frac{a}{a'} \phi \pmod{1}$ . A real number of the form  $\frac{b + \sqrt{D}}{2a}$ , where  $c = \frac{D - b^2}{4a}$  is an integer and GCD(a, b, c) = 1 is called a quadratic irrationality of discriminant D.