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by Proposition 3, we have $J = \rho I$, where $\psi = \frac{p\phi + q}{r\phi + s}$, $\rho = \frac{A}{a} \frac{1}{r\phi + s} = \varepsilon(r\bar{\phi} + s)$ and $\varepsilon = ps - qr = \pm 1$. Clearly we have

$$A = \varepsilon a(r\phi + s)(r\bar{\phi} + s) = \varepsilon(as^2 + bsr - cr^2),$$

$$\begin{aligned} B &= A(\psi + \bar{\psi}) = \varepsilon a(\psi + \bar{\psi})(r\phi + s)(r\bar{\phi} + s) \\ &= \varepsilon a((p\phi + q)(r\bar{\phi} + s) + (p\bar{\phi} + q)(r\phi + s)) \\ &= \varepsilon(2asq + b(sp + rq) - 2cpr), \end{aligned}$$

$$\begin{aligned} -C &= A\psi\bar{\psi} = \varepsilon a\psi\bar{\psi}(r\phi + s)(r\bar{\phi} + s) = \varepsilon a(p\phi + q)(p\bar{\phi} + q) \\ &= \varepsilon(aq^2 + bqp - cp^2). \end{aligned}$$

Thus A, B, C are integral linear combinations of a, b, c . Similarly, a, b, c are integral linear combinations of A, B, C . Hence $GCD(A, B, C) = GCD(a, b, c) = 1$ so that J is primitive.

3. THE HOMOMORPHISM θ

Let O_D and $O_{D'}$ be two orders of O_{D_0} with $O_{D'} \subset O_D$. Then we have $D' = Df^2$ for some positive integer f . This notation will be used throughout the rest of the paper. Our aim is to define a surjective homomorphism θ from the ideal class group $C_{D'}$ onto the ideal class group C_D . After proving three lemmas, we will prove the following theorem.

THEOREM 1. (i) Every class C of $C_{D'}$ contains a primitive ideal I of the form $I = \left[a, \frac{fb + \sqrt{D'}}{2} \right]$, where $GCD(a, f) = 1$, such that the ideal $J = \left[a, \frac{b + \sqrt{D}}{2} \right]$ is a primitive ideal of O_D .

(ii) If $I = \left[a, \frac{fb + \sqrt{D'}}{2} \right]$ ($GCD(a, f) = 1$) and $I' = \left[a', \frac{fb' + \sqrt{D'}}{2} \right]$ ($GCD(a', f) = 1$) are two primitive ideals in the same class C of $C_{D'}$ with $I' = \rho I$ ($\rho \in K^*$), then the ideals

$$J = \left[a, \frac{b + \sqrt{D}}{2} \right] \quad \text{and} \quad J' = \left[a', \frac{b' + \sqrt{D}}{2} \right]$$

of O_D satisfy $J' = \rho J$ and are in the same class $\theta(C)$ of C_D .

(iii) The mapping $C \rightarrow \theta(C)$ is a homomorphism of $C_{D'}$ to on C_D .

Part (ii) of Theorem 1 will be the main tool in relating distances between ideals of different orders of the same real quadratic field.

LEMMA 1. A primitive ideal $I = \left[a, \frac{b + \sqrt{D}}{2} \right]$ contains a number

$\alpha = xa + y \left(\frac{b + \sqrt{D}}{2} \right)$, where x and y are coprime integers, such that the integer $N(\alpha)/a$ is prime to a given nonzero integer m .

Proof. We begin by noting that $\frac{1}{a} N \left(xa + y \left(\frac{b + \sqrt{D}}{2} \right) \right) = ax^2 + bxy - cy^2$ in view of (2.5). If $|m| = 1$ we take $x = 1, y = 0, \alpha = xa + y \left(\frac{b + \sqrt{D}}{2} \right) = a$, so that $GCD(N(\alpha)/a, m) = GCD(a, 1) = 1$, as required.

Hence we may suppose that $|m| > 1$. Let $p_i (i = 1, 2, \dots, n)$ be the distinct prime factors of m . For $i = 1, 2, \dots, n$ we set

$$(x_i, y_i) = \begin{cases} (1, 0), & \text{if } p_i \nmid a, \\ (0, 1), & \text{if } p_i \mid a, \quad p_i \nmid c, \\ (1, 1), & \text{if } p_i \mid a, \quad p_i \mid c, \end{cases}$$

so that $p_i \nmid ax_i^2 + bx_iy_i - cy_i^2$. Let x' and y' be integers such that $x' \equiv x_i \pmod{p_i}$ and $y' \equiv y_i \pmod{p_i}$ for $i = 1, 2, \dots, n$, so that $GCD(ax'^2 + bx'y' - cy'^2, m) = 1$. The required number α is given by $\alpha = xa + y \left(\frac{b + \sqrt{D}}{2} \right)$, where $x = \frac{x'}{GCD(x', y')}$, $y = \frac{y'}{GCD(x', y')}$.

LEMMA 2. Let m be a given nonzero integer. Every class C of C_D contains a primitive ideal $\left[a, \frac{b + \sqrt{D}}{2} \right]$ with $GCD(a, m) = 1$.

Proof. Let $\left[a', \frac{b' + \sqrt{D}}{2} \right]$ be a primitive ideal of the class C . By Lemma 1 there exist coprime integers x and y such that

$$(3.1) \quad GCD(a'x^2 + b'xy - c'y^2, m) = 1.$$

Set $a = a'x^2 + b'xy - c'y^2$ and let r and s be integers such that $xs - yr = 1$. Next set

$$(3.2) \quad \rho = x + \left(\frac{b' - \sqrt{D}}{2a'} \right) y, \quad b = 2a'xr + b'(xs + yr) - 2c'ys,$$

so that

$$a = \rho \left(xa' + y \left(\frac{b' + \sqrt{D}}{2} \right) \right)$$

and

$$\frac{b + \sqrt{D}}{2} = \rho \left(ra' + s \left(\frac{b' + \sqrt{D}}{2} \right) \right).$$

Then we have

$$\begin{aligned} \left[a, \frac{b + \sqrt{D}}{2} \right] &= \rho \left[xa' + y \left(\frac{b' + \sqrt{D}}{2} \right), ra' + s \left(\frac{b' + \sqrt{D}}{2} \right) \right] \\ &= \rho \left[a', \frac{b' + \sqrt{D}}{2} \right] \end{aligned}$$

so that $\left[a, \frac{b + \sqrt{D}}{2} \right]$ is an ideal equivalent to the primitive ideal $\left[a', \frac{b' + \sqrt{D}}{2} \right]$. Hence, by Corollary 3, $\left[a, \frac{b + \sqrt{D}}{2} \right]$ is primitive.

LEMMA 3. *Let C and C' be two classes of C_D . Then there exist primitive ideals $I = \left[a, \frac{B + \sqrt{D}}{2} \right] \in C$ and $I' = \left[a', \frac{B + \sqrt{D}}{2} \right] \in C'$ with $\text{GCD}(a, a') = 1$. Moreover the ideal II' is primitive and $II' = \left[aa', \frac{B + \sqrt{D}}{2} \right]$.*

Proof. By Lemma 2 there exist primitive ideals $I = \left[a, \frac{b + \sqrt{D}}{2} \right] \in C$ and $I' = \left[a', \frac{b' + \sqrt{D}}{2} \right] \in C'$ with $\text{GCD}(a, a') = 1$. As $b \equiv D \equiv b' \pmod{2}$ and $\text{GCD}(a, a') = 1$ there are integers k and k' such that $k'a' - ka = \frac{b - b'}{2}$.

Set $B = b + 2ka = b' + 2k'a'$ so that

$$I = \left[a, \frac{B + \sqrt{D}}{2} \right] \quad \text{and} \quad I' = \left[a', \frac{B + \sqrt{D}}{2} \right].$$

Now $D - B^2$ is divisible by both $4a$ and $4a'$, and so, as $GCD(a, a') = 1$, $D - B^2$ is a multiple of $4aa'$, so that $c'' = \frac{D - B^2}{4aa'} \in Z$. Hence $\left[aa', \frac{B + \sqrt{D}}{2} \right]$

is an ideal of O_D and we have

$$\begin{aligned} II' &= \left(aa', a \left(\frac{B + \sqrt{D}}{2} \right), a' \left(\frac{B + \sqrt{D}}{2} \right), \left(\frac{B + \sqrt{D}}{2} \right)^2 \right) \\ &= \left(aa', \frac{B + \sqrt{D}}{2} \right) \\ &= \left[aa', \frac{B + \sqrt{D}}{2} \right]. \end{aligned}$$

Finally, any prime divisor of aa', B, c'' must divide $GCD(a, B, a'c'') = 1$ or $GCD(a', B, ac'') = 1$, as $GCD(a, a') = 1$, which is impossible. Hence the ideal II' is primitive.

We are now ready to prove Theorem 1.

Proof of Theorem 1. (i) By Lemma 2 the class C contains a primitive ideal $I = \left[a, \frac{b' + \sqrt{D'}}{2} \right]$ with $GCD(a, f) = 1$. Let k be an integer such that

$$\begin{cases} 2ak \equiv -b' \pmod{f}, & \text{if } f \equiv 1 \pmod{2}, \\ ak \equiv -\frac{b'}{2} + D \frac{f}{2} \pmod{f}, & \text{if } f \equiv 0 \pmod{2}, \end{cases}$$

and set $b = (b' + 2ak)/f$, so that $I = \left[a, \frac{fb + \sqrt{D'}}{2} \right]$. As I is an ideal of $O_{D'}$, $(D' - f^2b^2)/4a$ is an integer, and so, as $GCD(a, f) = 1$, $c = (D - b^2)/4a$ is also an integer, showing that $J = \left[a, \frac{b + \sqrt{D}}{2} \right]$ is an ideal of O_D . Further, as I is primitive, we have $GCD(a, bf, cf^2) = 1$, and so $GCD(a, b, c) = 1$, showing that J is primitive.

(ii) If $I' = \rho I$, by Proposition 3, there exist integers p, q, r, s with $ps - qr = \pm 1$ such that

$$(3.3) \quad \frac{fb' + \sqrt{D'}}{2a'} = \frac{p \left(\frac{fb + \sqrt{D'}}{2a} \right) + q}{r \left(\frac{fb + \sqrt{D'}}{2a} \right) + s}, \quad \rho = \pm \left(r \left(\frac{fb - \sqrt{D'}}{2a} \right) + s \right).$$

Rearranging the first equation in (3.3), we obtain the following equality among elements of O_D

$$f\left(\frac{b' + \sqrt{D}}{2}\right) \left(rf\left(\frac{b + \sqrt{D}}{2}\right) + sa \right) = a' \left(pf\left(\frac{b + \sqrt{D}}{2}\right) + qa \right),$$

from which we deduce that $f \mid qaa'$. As $GCD(aa', f) = 1$ there exists an integer q' such that $q = q'f$, so (3.3) can be rewritten as

$$\frac{b' + \sqrt{D}}{2a'} = \frac{p \left(\frac{b + \sqrt{D}}{2a} \right) + q'}{rf\left(\frac{b + \sqrt{D}}{2a}\right) + s}, \quad \rho = \pm \left(rf\left(\frac{b - \sqrt{D}}{2a}\right) + s \right).$$

which, by Proposition 3, shows that $J' = \rho J$.

(iii) Let $C \in C_{D'}$ and $C' \in C_{D'}$. By Lemma 2 and (i), we can choose an ideal $I = \left[a, f\left(\frac{b + \sqrt{D}}{2}\right) \right]$ in C with $GCD(a, f) = 1$ and then an ideal $I' = \left[a', f\left(\frac{b' + \sqrt{D}}{2}\right) \right]$ in C' with $GCD(a', af) = 1$. By (i) $\left[a, \frac{b + \sqrt{D}}{2} \right]$ and $\left[a', \frac{b' + \sqrt{D}}{2} \right]$ are ideals of O_D and so we have $b \equiv b' \pmod{2}$. We

choose integers K' and K such that $K'a' - Ka = \frac{b - b'}{2}$, and set $B = b + 2Ka$

$= b' + 2K'a'$, so that $I = \left[a, f\left(\frac{B + \sqrt{D}}{2}\right) \right]$ and $I' = \left[a', f\left(\frac{B + \sqrt{D}}{2}\right) \right]$.

By Lemma 3 we see that $II' = \left[aa', f\left(\frac{B + \sqrt{D}}{2}\right) \right]$ is a primitive ideal of the

class CC' . But the primitive ideals $J = \left[a, \frac{B + \sqrt{D}}{2} \right]$, $J' = \left[a', \frac{B + \sqrt{D}}{2} \right]$,

$J'' = \left[aa', \frac{B + \sqrt{D}}{2} \right]$ belong respectively to the classes $\theta(C)$, $\theta(C')$, $\theta(CC')$,

and, as $JJ' = J''$ by Lemma 3, we have $\theta(C)\theta(C') = \theta(CC')$, showing that θ is a homomorphism: $C_{D'} \rightarrow C_D$.

Finally we show that θ is surjective. Let C be a class of C_D and let $J = \left[a, \frac{b + \sqrt{D}}{2} \right]$ be a primitive ideal of C with $GCD(a, f) = 1$ (Lemma 2).

Then we have $GCD(a, b, c) = 1$, where $\frac{D - b^2}{4a} = c$, and so $GCD(a, bf, cf^2) = 1$,

showing that $I = \left[a, f \left(\frac{b + \sqrt{D}}{2} \right) \right]$ is a primitive ideal of $O_{D'}$. Hence C is the image of the class of I under θ .

COROLLARY 4. *If the class C of $O_{D'}$ contains the primitive ideal $I = \left[a, \frac{b + \sqrt{D'}}{2} \right]$, where $f^2 | a$, then $f | b$ and the class $\theta(C)$*

contains the primitive ideal $J = \left[\frac{a}{f^2}, \frac{\frac{b}{f} + \sqrt{D}}{2} \right]$ of O_D .

Proof. As $D' = Df^2 = b^2 + 4ac$, and $f^2 | a$, we see that $f | b$, and so $GCD(f, c) = 1$. By Corollary 2 we have $I = \left(\frac{\sqrt{D'} - b}{2a} \right) \left[c, \frac{-b + \sqrt{D'}}{2} \right]$

and so, by Theorem 1, we see that $\left[c, \frac{-\frac{b}{f} + \sqrt{D}}{2} \right] \in \theta(C)$. Finally,

by Corollary 2, $J = \left[\frac{a}{f^2}, \frac{b/f + \sqrt{D}}{2} \right] = \frac{\left(\sqrt{D} + \frac{b}{f} \right)}{2c} \left[c, \frac{-\frac{b}{f} + \sqrt{D}}{2} \right]$,

showing that $J \in \theta(C)$.

4. REDUCED IDEALS

From now on in this paper we suppose that $D_0 > 0$ so that we are only considering ideals in orders of a real quadratic field. An ideal I of O_D can be written in the form $I = ad[1, \phi]$, where $\phi = \frac{b + \sqrt{D}}{2a}$. By Proposition 1 (ii), if $I = a'd'[1, \phi']$ is another representation of I , then $a' = \pm a$ and $\phi' \equiv \frac{a}{a'} \phi \pmod{1}$. A real number of the form $\frac{b + \sqrt{D}}{2a}$, where $c = \frac{D - b^2}{4a}$ is an integer and $GCD(a, b, c) = 1$ is called a quadratic irrationality of discriminant D .