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Throughout this paper, if A is a unitary commutative ring, and $\alpha_1, \alpha_2, \dots, \alpha_m$ are elements of A , the \mathbb{Z} -module generated by $\alpha_1, \alpha_2, \dots, \alpha_m$ is denoted by $[\alpha_1, \alpha_2, \dots, \alpha_m]$ and the A -module (ideal) generated by $\alpha_1, \alpha_2, \dots, \alpha_m$ by $(\alpha_1, \alpha_2, \dots, \alpha_m)$. The product of the ideals $(\alpha_1, \dots, \alpha_m)$ and $(\alpha'_1, \dots, \alpha'_n)$ is the ideal $(\alpha_1 \alpha'_1, \dots, \alpha_i \alpha'_j, \dots, \alpha_m \alpha'_r)$. If I is an ideal, we often write the product ideal $(\alpha)I$ as αI .

2. BASIC DEFINITIONS

Let K be a quadratic field of discriminant D_0 . As D_0 is a discriminant we have $D_0 \equiv 0 \pmod{4}$ or $D_0 \equiv 1 \pmod{4}$. In §2 and §3 K may be real ($D_0 > 0$) or imaginary ($D_0 < 0$) but in the remaining sections K will be assumed to be real. An element α of K can be written $\alpha = x + y\sqrt{D_0}$, where x and y are rational numbers. The conjugate of α is the element $\bar{\alpha} = x - y\sqrt{D_0}$ of K . The norm of α is the rational number $N(\alpha) = \alpha\bar{\alpha} = x^2 - D_0y^2$. We define the integer ω_0 of K by

$$(2.1) \quad \omega_0 = \begin{cases} \frac{\sqrt{D_0}}{2}, & \text{if } D_0 \equiv 0 \pmod{4}, \\ \frac{1}{2}(1 + \sqrt{D_0}), & \text{if } D_0 \equiv 1 \pmod{4}. \end{cases}$$

The ring of integers of K is $O_{D_0} = [1, \omega_0]$. For a positive integer f , we set

$$(2.2) \quad D = D_0f^2, \omega = \begin{cases} \frac{\sqrt{D}}{2}, & \text{if } D \equiv 0 \pmod{4} \\ \frac{1}{2}(1 + \sqrt{D}), & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

and

$$(2.3) \quad O_D = [1, \omega] = [1, f\omega_0].$$

It is easy to check that O_D is the subring of index f in O_{D_0} , called the order of discriminant D . We note that

$$(2.4) \quad \omega^2 = \begin{cases} \frac{D}{4}, & \text{if } D \equiv 0 \pmod{4}, \\ \omega + \frac{(D-1)}{4}, & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

The multiplicative group of K is denoted by K^* .

Next we describe the ideals of the order O_D . Throughout this paper all ideals will be nonzero.

PROPOSITION 1. ([10]: Theorem 5.6, [12]: Theorem 3.2) (i) *The (nonzero) ideals of the order O_D are the Z -modules*

$$I = d \left[a, \frac{b + \sqrt{D}}{2} \right],$$

where

$$(2.5) \quad c = \frac{D - b^2}{4a}$$

is an integer.

(ii) *Two ideals $I = d \left[a, \frac{b + \sqrt{D}}{2} \right]$ and $I' = d' \left[a', \frac{b' + \sqrt{D}}{2} \right]$ are equal if, and only if, $|d| = |d'|, |a| = |a'|, b \equiv b' \pmod{2a}$.*

Proof. (i) Let I be a (nonzero) ideal of O_D . The set $I \cap Z$ is a (nonzero) ideal (a_0) of Z . The set $\{y \in Z: x + y\omega \in I \text{ for some } x \in Z\}$ is also an ideal (d) of Z , and, as $a_0\omega \in I$, we see that $d|a_0$, say $a_0 = da$. Let $\alpha_0 \in I$ be such that $\alpha_0 = b_0 + d\omega$. Appealing to (2.4), we see that

$$\omega\alpha_0 = \omega(b_0 + d\omega) = \begin{cases} \frac{dD}{4} + b_0\omega, & \text{if } D \equiv 0 \pmod{4}, \\ d \left(\frac{D-1}{4} \right) + (d + b_0)\omega, & \text{if } D \equiv 1 \pmod{4}, \end{cases}$$

so that $d|b_0$, say $b_0 = db_1$. Thus we have $\alpha_0 = d(b_1 + \omega)$, which shows that $I \supseteq d[a, b_1 + \omega]$. Now let $\beta = x + dy\omega \in I$. As $\beta - \alpha_0 y = x - b_0 y \in I \cap Z$, there exists $k \in Z$ such that $\beta = ka_0 + \alpha_0 y$, which shows that $I \subseteq [a_0, \alpha_0] = d[a, b_1 + \omega]$. Hence we have $I = d[a, b_1 + \omega]$. As $dN(b_1 + \omega) = d(b_1 + \omega)(b_1 + \bar{\omega}) \in I \cap Z = (da)$, we see that a divides $N(b_1 + \omega)$.

Now let $I = d[a, b_1 + \omega]$, where $c = -N(b_1 + \omega)|a$ is an integer. We show that I is an ideal of O_D . It suffices to prove that ωa and $\omega(b_1 + \omega)$ belong to $[a, b_1 + \omega]$. This follows from

$$\omega a = (-b_1)a + a(b_1 + \omega)$$

and

$$\begin{aligned}\omega(b_1 + \omega) &= - (b_1 + \bar{\omega}) (b_1 + \omega) + (b_1 + \omega + \bar{\omega}) (b_1 + \omega) \\ &= ca + (b_1 + \omega + \bar{\omega}) (b_1 + \omega) .\end{aligned}$$

We have thus shown that the ideals of O_D are the Z -modules $d[a, b_1 + \omega]$, where $c = -N(b_1 + \omega)|a$ is an integer. Let b be the integer given by

$$b = \begin{cases} 2b_1, & \text{if } D \equiv 0 \pmod{3}, \\ 2b_1 + 1, & \text{if } D \equiv 1 \pmod{4}, \end{cases}$$

so that

$$b_1 + \omega = \frac{b + \sqrt{D}}{2}, \quad \frac{N(b_1 + \omega)}{a} = \frac{b^2 - D}{4a} = -c \in Z .$$

This completes the proof of Proposition 1 (i).

(ii) If $d \left[a, \frac{b + \sqrt{D}}{2} \right] = d' \left[a', \frac{b' + \sqrt{D}}{2} \right]$ we easily see that $d|d'$, $d'|d$, $ad|a'd'$ and $a'd'|ad$, from which Proposition 1 (ii) follows.

Example 1. (i) By Proposition 1 (i) the Z -module $A = \left[3, \frac{1 + \sqrt{45}}{2} \right]$ of O_{45} is not an ideal of O_{45} as $\frac{45 - 1}{12}$ is not an integer. Indeed A is not closed under multiplication by elements of O_{45} as $\frac{1 + \sqrt{45}}{2} \in A$ but

$$\left(\frac{1 - \sqrt{45}}{2} \right) \left(\frac{1 + \sqrt{45}}{2} \right) = -11 \notin A .$$

(ii) By Proposition 1 (i) the Z -module $B = \left[11, \frac{1 + \sqrt{45}}{2} \right]$ of O_{45} is an ideal of O_{45} as $\frac{45 - 1}{44}$ is an integer.

If $I = d \left[a, \frac{b + \sqrt{D}}{2} \right]$ is an ideal of O_D , by Proposition 1 (ii), we see that $GCD(a, b, c)$ does not depend upon the choice of a, b and d . This enables us to define the concept of a primitive ideal of O_D .

Definition 1. (Primitive ideal) The ideal $I = d \left[a, \frac{b + \sqrt{D}}{2} \right]$ of O_D is called *primitive* if, and only if,

$$d = \text{GCD}(a, b, c) = 1 ,$$

where c is defined by (2.5).

Our next result gives some basic properties of primitive ideals.

PROPOSITION 2. ([10]: Theorem 5.9) (i) If $I = \left[a, \frac{b + \sqrt{D}}{2} \right]$ is a primitive ideal of O_D then

$$I\bar{I} = (a) ,$$

where $\bar{I} = \left[a, \frac{b - \sqrt{D}}{2} \right]$ is the conjugate ideal of I .

(ii) If I is a primitive ideal of O_D and $\alpha \in K^*$ is such that $I = \alpha I$, then α is a unit of O_D .

(iii) If $I = \left[a, \frac{b + \sqrt{D}}{2} \right]$ and $J = \left[A, \frac{B + \sqrt{D}}{2} \right]$ are primitive ideals of O_D such that $\frac{1}{a}I = \frac{1}{A}J$ then $I = J$ and $|a| = |A|$.

Proof. (i) We have

$$I\bar{I} = a \left(a, \frac{b + \sqrt{D}}{2}, \frac{b - \sqrt{D}}{2}, c \right) .$$

The ideal $\left(a, \frac{b + \sqrt{D}}{2}, \frac{b - \sqrt{D}}{2}, c \right)$ contains the ideal $(a, b, c) = (1)$, so that $I\bar{I} = (a)$.

(ii) As $\alpha \in K^*$, there exist $\beta \in O_D^*$ and $\gamma \in O_D^*$ such that $\alpha = \beta / \gamma$. Then, we have $\gamma I = \gamma \alpha I = \beta I$, and so, by (i), we obtain $(\gamma)(a) = \gamma I\bar{I} = \beta I\bar{I} = (\beta)(a)$, giving $(\beta) = (\gamma)$, so that $\alpha = \beta / \gamma$ is a unit of O_D .

(iii) We have $AI = aJ$ so that, by (ii), $a/A = \pm 1$ and $I = J$.

Next we define the notion of equivalent ideals.

Definition 2. (Equivalent ideals) Two ideals I and I' of O_D are said to be *equivalent* if there exists $\rho \in K^*$ such that $I' = \rho I$.

Example 2. The ideals

$$I = \left[7, \frac{12 + \sqrt{200}}{2} \right] = [7, 6 + \sqrt{50}] \quad \text{and} \quad J = \left[2, \frac{\sqrt{200}}{2} \right] = [2, \sqrt{50}]$$

of O_{200} are equivalent as

$$\begin{aligned} I &= [7, -8 + \sqrt{50}] \\ &= \left(\frac{-8 + \sqrt{50}}{2} \right) [-8 - \sqrt{50}, 2] \\ &= \left(\frac{-16 + \sqrt{200}}{4} \right) [2, \sqrt{50}] \\ &= \alpha J, \end{aligned}$$

where

$$\alpha = \frac{-16 + \sqrt{200}}{4} \in K^* .$$

It is clear that the notion of equivalence given in Definition 2 is an equivalence relation. The equivalence classes are called ideal classes. The ideal class of the ideal I is denoted by $C(I)$. If $I' \in C(I)$ and $J' \in C(J)$ then $I'J' \in C(IJ)$, and we can define multiplication of ideal classes by $C(I)C(J) = C(IJ)$.

Definition 3. (Primitive class) An ideal class of O_D containing a primitive ideal is called a *primitive class*.

It follows from Proposition 2(i) that the primitive classes are invertible, and so form a group C_D with respect to multiplication.

Definition 4. (Ideal class group) The group C_D of primitive classes of the order O_D is called the *ideal class group* of O_D .

The unit class of the ideal class group is called the principal class and consists of all the principal primitive ideals of O_D . In fact C_D is a finite group.

Next we give a necessary and sufficient condition for two ideals I and I' of O_D to be equivalent, and, when I and I' are equivalent, a means of calculating ρ in the relationship $I' = \rho I$. It suffices to consider ideals of the form $\left[a, \frac{b + \sqrt{D}}{2} \right]$ that is with $d = 1$.

PROPOSITION 3. ([10]: Theorem 5.27) *Let*

$$I = \left[a, \frac{b + \sqrt{D}}{2} \right] \quad \text{and} \quad J = \left[A, \frac{B + \sqrt{D}}{2} \right]$$

be two ideals of O_D . *Set*

$$\phi = \frac{b + \sqrt{D}}{2a}, \quad \psi = \frac{B + \sqrt{D}}{2A}.$$

(i) *The ideals* I *and* J *are equivalent if, and only if, there exists a* 2×2 *integral matrix* $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ *of determinant* $\varepsilon = ps - qr = \pm 1$ *such that*

$$\psi = \frac{p\phi + q}{r\phi + s}.$$

(ii) *If* I *and* J *are equivalent the numbers* $\rho \in K^*$ *such that* $J = \rho I$ *are given by*

$$(2.6) \quad \rho = \frac{A}{a} \frac{1}{r\phi + s} = \varepsilon \frac{1}{r\bar{\phi} + s}$$

and satisfy

$$(2.7) \quad N(\rho) = \varepsilon \frac{A}{a}.$$

Proof. We have $J = \rho I$, that is $A[1, \psi] = \rho a[1, \phi]$, if, and only if, there exists an integral matrix $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ of determinant $\varepsilon = \pm 1$ such that

$$(2.8) \quad \begin{cases} A = r\rho a\phi + s\rho a, \\ A\psi = p\rho a\phi + q\rho a. \end{cases}$$

The equations (2.8) are equivalent to

$$\psi = \frac{p\phi + q}{r\phi + s}, \quad \rho = \frac{A}{a} \frac{1}{r\phi + s}.$$

This establishes (i) and the first equality of (2.6).

Taking conjugates in (2.8), we have

$$(2.9) \quad \begin{cases} A = r\bar{\rho}a\bar{\phi} + s\bar{\rho}a, \\ A\bar{\psi} = p\bar{\rho}a\bar{\phi} + q\bar{\rho}a, \end{cases}$$

so that (2.8) and (2.9) are equivalent to the matrix equality

$$\begin{bmatrix} A\psi & A \\ A\bar{\psi} & A \end{bmatrix} = \begin{bmatrix} a\phi\rho & a\rho \\ a\bar{\phi}\bar{\rho} & a\bar{\rho} \end{bmatrix} \begin{bmatrix} p & r \\ q & s \end{bmatrix}.$$

Taking determinants we obtain

$$A^2(\psi - \bar{\psi}) = \varepsilon\rho\bar{\rho}a^2(\phi - \bar{\phi}),$$

which gives, as $\psi - \bar{\psi} = \frac{\sqrt{D}}{A}$ and $\phi - \bar{\phi} = \frac{\sqrt{D}}{a}$, $\rho\bar{\rho} = \varepsilon \frac{A}{a}$, proving (2.7).

Then the first equality in (2.6) shows that $\bar{\rho} = \varepsilon(r\phi + s)$, establishing the second equality in (2.6).

COROLLARY 1. Let $I = \left[a, \frac{b + \sqrt{D}}{2} \right]$ be a primitive ideal of O_D , and set $\phi = \frac{b + \sqrt{D}}{2a}$. For $q \in \mathbb{Z}$ define ϕ', b', a' and I' by

(2.10)

$$\phi = q + \frac{1}{\phi'}, \quad b' = -b + 2aq, \quad a' = \frac{D - b'^2}{4a}, \quad I' = \left[a', \frac{b' + \sqrt{D}}{2} \right].$$

Then

$$(2.11) \quad a' = \frac{D - b^2}{4a} + bq - aq^2 \in \mathbb{Z}, \quad \phi' = \frac{b' + \sqrt{D}}{2a'},$$

and I' is a primitive ideal of O_D such that

$$(2.12) \quad I' = \frac{a'}{a} \phi' I = \frac{-1}{\bar{\phi}'} I.$$

Proof. The formulas in (2.11) for a' and ϕ' are easily proved by a straightforward calculation, and Proposition 3 with $p = 0$, $q = 1$, $r = 1$, $s = -q$ gives

$$I' = \frac{a'}{a} \frac{1}{\phi - q} I = -(\bar{\phi} - q)I,$$

which is equivalent to (2.12) as $\phi' = \frac{1}{\phi - q}$.

By Proposition 1 a primitive ideal I of O_D can be written in the form $I = a[1, \phi]$ ($\phi = (b + \sqrt{D})/2a$), where a is an integer uniquely determined up to sign by I and $a\phi$ is determined modulo a by I .

Definition 5. (Representation of a primitive ideal). Let I be a primitive ideal of O_D . A pair $\{a, b\}$ such that $I = a[1, \phi]$, where $\phi = (b + \sqrt{D})/2a$, is called a *representation* of I .

Definition 6. (q -neighbour). When the representation $\{a, b\}$ of the ideal I and the representation $\{a', b'\}$ of the ideal I' are related as in (2.10), we say that $\{a', b'\}$ is q -neighbour to $\{a, b\}$.

Definition 7. (Lagrange neighbour). When $D > 0$ and $\{a', b'\}$ is q -neighbour to $\{a, b\}$ with $q = [\phi]$, we say that $\{a', b'\}$ is the *Lagrange neighbour* of $\{a, b\}$ and write $\{a, b\} \xrightarrow{L} \{a', b'\}$.

Definition 8. (Gauss neighbour). When $D > 0$ and $\{a', b'\}$ is q -neighbour to $\{a, b\}$ with $q = \frac{a}{|a|} \left[\frac{a}{|a|} \phi \right]$, we say that $\{a', b'\}$ is the *Gauss neighbour* of $\{a, b\}$ and write $\{a, b\} \xrightarrow{G} \{a', b'\}$.

Lagrange's reduction process using Lagrange neighbours is described in §5 and Gauss's reduction process using Gauss neighbours in §8.

COROLLARY 2. The ideals $I = \left[a, \frac{b + \sqrt{D}}{2} \right]$ and $J = \left[c, \frac{-b + \sqrt{D}}{2} \right]$, where c is given by (2.5), are equivalent and satisfy

$$J = \frac{(-b + \sqrt{D})}{2a} I.$$

Proof. We have $\psi = \frac{1}{\phi}$, where $\phi = \frac{b + \sqrt{D}}{2a}$ and $\psi = \frac{-b + \sqrt{D}}{2c}$, so that, by Proposition 3(ii), we have $J = \rho I$ with $\rho = (-1)\bar{\phi} = \frac{-b + \sqrt{D}}{2a}$.

COROLLARY 3. If $I = \left[a, \frac{b + \sqrt{D}}{2} \right]$ and $J = \left[A, \frac{B + \sqrt{D}}{2} \right]$ are two equivalent ideals of O_D with I primitive then J is also primitive.

Proof. Set $\phi = \frac{b + \sqrt{D}}{2a}$ and $\psi = \frac{B + \sqrt{D}}{2A}$. As I and J are equivalent,

by Proposition 3, we have $J = \rho I$, where $\psi = \frac{p\phi + q}{r\phi + s}$, $\rho = \frac{A}{a} \frac{1}{r\phi + s} = \varepsilon(r\bar{\phi} + s)$ and $\varepsilon = ps - qr = \pm 1$. Clearly we have

$$A = \varepsilon a(r\phi + s)(r\bar{\phi} + s) = \varepsilon(as^2 + bsr - cr^2),$$

$$\begin{aligned} B &= A(\psi + \bar{\psi}) = \varepsilon a(\psi + \bar{\psi})(r\phi + s)(r\bar{\phi} + s) \\ &= \varepsilon a((p\phi + q)(r\bar{\phi} + s) + (p\bar{\phi} + q)(r\phi + s)) \\ &= \varepsilon(2asq + b(sp + rq) - 2cpr), \end{aligned}$$

$$\begin{aligned} -C &= A\psi\bar{\psi} = \varepsilon a\psi\bar{\psi}(r\phi + s)(r\bar{\phi} + s) = \varepsilon a(p\phi + q)(p\bar{\phi} + q) \\ &= \varepsilon(aq^2 + bqp - cp^2). \end{aligned}$$

Thus A, B, C are integral linear combinations of a, b, c . Similarly, a, b, c are integral linear combinations of A, B, C . Hence $GCD(A, B, C) = GCD(a, b, c) = 1$ so that J is primitive.

3. THE HOMOMORPHISM θ

Let O_D and $O_{D'}$ be two orders of O_{D_0} with $O_{D'} \subset O_D$. Then we have $D' = Df^2$ for some positive integer f . This notation will be used throughout the rest of the paper. Our aim is to define a surjective homomorphism θ from the ideal class group $C_{D'}$ onto the ideal class group C_D . After proving three lemmas, we will prove the following theorem.

THEOREM 1. (i) Every class C of $C_{D'}$ contains a primitive ideal I of the form $I = \left[a, \frac{fb + \sqrt{D'}}{2} \right]$, where $GCD(a, f) = 1$, such that the ideal $J = \left[a, \frac{b + \sqrt{D}}{2} \right]$ is a primitive ideal of O_D .

(ii) If $I = \left[a, \frac{fb + \sqrt{D'}}{2} \right]$ ($GCD(a, f) = 1$) and $I' = \left[a', \frac{fb' + \sqrt{D'}}{2} \right]$ ($GCD(a', f) = 1$) are two primitive ideals in the same class C of $C_{D'}$ with $I' = \rho I$ ($\rho \in K^*$), then the ideals

$$J = \left[a, \frac{b + \sqrt{D}}{2} \right] \quad \text{and} \quad J' = \left[a', \frac{b' + \sqrt{D}}{2} \right]$$

of O_D satisfy $J' = \rho J$ and are in the same class $\theta(C)$ of C_D .