3. Proof of Theorem 1

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If $\rho = \min\{r/2, \pi/4\sqrt{K}\}$, then $P(\rho)$ holds, by taking $\delta = \min\{\varepsilon/2, \rho/2\}$. This estimate uses the the fact that the distances between corresponding points of two sides of a triangle in S_K never exceed the endpoint value if both sides have length less than $\pi/2\sqrt{K}$. It remains to be shown that if $P(L)$ holds and $L < 2\pi/3\sqrt{K}$, then $P(3L/2)$ holds.

Choose $\varepsilon < \min\{r/2, \pi/6\sqrt{K}\}$. This choice of ε ensures that any two geodesics issuing from the same point and having length at most $L + \varepsilon$ and distance at most ε from a subsegment of γ will satisfy the uniform distance comparison property. In particular, the geodesics $\alpha(p, q)$ in $P(L)$ are unique. Now choose $\varepsilon' < \min\{2\pi/3\sqrt{K} - L, 2\varepsilon/3\}$. Denote by δ' the value given by applying $P(L)$ to γ, with ε' as the desired distance from subsegments of γ. Set $L' = L + \varepsilon'$ and $\lambda = \sin(\sqrt{K}L'/2)/\sin(\sqrt{K}L')$ (then $1/2 < \lambda < 1$). It is an exercise in spherical trigonometry to show that if β_1 and β_2 are two sides of a minimizing triangle in S_K and both have length less than L', then

$$d(\beta_1(1/2), \beta_2(1/2)) < \lambda d(\beta_1(1), \beta_2(1)) \,. \tag{1}$$

Let $\delta = (1 - \lambda)\delta'/\lambda$ (then also $\delta < \delta'$).

Suppose that $\bar{\gamma}$ is a subsegment of γ of length $\bar{L} \leqslant 3L/2$, with endpoints $\bar{p}$ and $\bar{q}$, and let p and q be points within distance δ of these endpoints. We now follow the construction of Theorem 2. Subdivide $\bar{\gamma}$ into thirds by points p_0, q_0 and take, recursively, p_i as the midpoint of $\alpha(p, q_{i-1})$ and q_i as the midpoint of $\alpha(p_{i-1}, q)$. To verify that this recursive definition is possible, apply $P(L)$ repeatedly to the subsegments $\alpha(\bar{p}, q_0)$ and $\alpha(p_0, \bar{q})$ of γ, and note that inductively $d(p_{i-1}, p_i)$ and $d(q_{i-1}, q_i)$ are less than $\lambda^i\delta$ by the uniform distance comparison property and (1), and hence $d(p_0, p_i)$ and $d(q_0, q_i)$ are less than $\lambda\delta/(1 - \lambda) = \delta'$. In particular, $\{p_i\}$ and $\{q_i\}$ are Cauchy, and converge to p_∞ and q_∞ respectively. By the uniform distance comparison property, $\{\alpha(p, q_i)\}$ converges uniformly to $\alpha(p, q_\infty)$ and $\{\alpha(p_i, q)\}$ to $\alpha(p_\infty, q)$. These two limit geodesics overlap since $\alpha(p_\infty, q_\infty)$ is unique, hence combine to give a geodesic from p to q that has distance at most $\varepsilon' < \varepsilon$ from $\bar{\gamma}$ and length at most $\bar{L} + 3\varepsilon'/2 < \bar{L} + \varepsilon$. □

3. Proof of Theorem 1

Again consider a locally convex, complete geodesic space M, and let $\mathbf{G}_m$ be the space of geodesics starting at m carrying the uniform metric $\mathbf{d}$. It follows from local convexity that a Cauchy sequence in $\mathbf{G}_m$ converges to a geodesic and hence $\mathbf{d}$ is complete. Furthermore, M has *neighborhoods of bipoint uni-*

queness, any two points of which are joined by a unique minimizing geodesic in M, varying continuously with its endpoints. Note that neighborhoods of bipoint uniqueness guarantee that each point is the center of a contractible metric ball, since a minimizing geodesic from the center of a metric ball lies in the ball if its righthand endpoint does. Thus any interior metric space with neighborhoods of bipoint uniqueness is pathconnected, locally pathconnected and locally simply connected, and covering space theory applies. We now give a proof of Theorem 1, restated as follows:

THEOREM 4. *In a locally convex, complete geodesic space, the endpoint map on the space of geodesics from any given point is the universal covering map. Thus each homotopy class of curves between two given points contains exactly one geodesic.*

If M is simply connected, it follows that the members of $\mathbf{G}_m$ are uniquely determined by and vary continuously with their righthand endpoints. (Thus M is contractible.) Furthermore, the distance function between any two geodesics in M is convex, as claimed by Theorem 1. Indeed, for this it suffices to verify the midpoint convexity property for two sides of an arbitrary geodesic triangle (see [Bu2], p. 237); by continuity, one can subdivide into arbitrarily thin triangles, for which convexity is obvious.

We use a covering lemma for interior (rather than geodesic) spaces. It will be applied to the case in which $\overline{M}$ is $\mathbf{G}_m$, and is not known to contain minimizing geodesics between pairs of its members. By saying ϕ is a *local isometry of* $\overline{M}$ *onto* M we mean that every point in $\overline{M}$ has a neighborhood mapped isometrically onto a neighborhood of the image point.

LEMMA 1. *Let* M *and* $\overline{M}$ *be complete interior metric spaces. If* M *has neighborhoods of bipoint uniqueness, then any local isometry* ϕ *of* $\overline{M}$ *onto* M *is a covering map.*

Proof. Choose the open metric ball $B(p, \varepsilon)$ to be a neighborhood of bipoint uniqueness in M. Since $\overline{M}$ is not necessarily geodesic, we argue as follows to show that the restriction of ϕ to $B(\bar{p}, \varepsilon)$ is injective, for $\bar{p}$ in the preimage of p. Since $\overline{M}$ is interior, any two points of $B(\bar{p}, \varepsilon)$ may be joined to $\bar{p}$ by curves $\bar{\alpha}$ and $\bar{\beta}$ which map $[0, 1]$ into $B(\bar{p}, \varepsilon)$. Their image curves, α and β, lie in $B(p, \varepsilon)$, since a local isometry preserves lengths and hence does not increase distances. There is a continuous variation of minimizing geodesics γ_t from $\alpha(t)$ to $\beta(t)$, $0 \leqslant t \leqslant 1$. Since a local isometry between complete spaces has the unique pathlifting property, one may lift $\alpha \mid [0, t]$ followed by

γ_t, for each t, to $\bar{p}$. This gives a continuous curve starting at $\bar{p}$ and lying over β, hence coinciding with $\bar{\beta}$. If α and β have the same righthand endpoint, then γ_1 is constant, hence $\bar{\alpha}$ and $\bar{\beta}$ also have the same righthand endpoint.

From here it is straightforward to check that the preimage of $B(p, \varepsilon)$ has the desired form. For instance, the fact that $B(\bar{p}_1, \varepsilon)$ and $B(\bar{p}_2, \varepsilon)$ are disjoint for distinct $\bar{p}_1$, $\bar{p}_2$ in the preimage of p has almost the same proof as above. □

Proof of Theorem 4. Note that $\mathbf{G}_m$ is contractible, hence connected and simply connected. By Lemma 1, it suffices to define a complete interior metric $\mathbf{d}^*$ on $\mathbf{G}_m$, with respect to which the endpoint map is a local isometry and whose topology agrees with that of $\mathbf{d}$. (In general $\mathbf{d}$ is not interior; for example, take M to be a Euclidean circle.) Let $\mathbf{d}^*$ be the interior metric induced by $\mathbf{d}$; that is, let $\mathbf{d}^*(\gamma, \sigma)$ be the infimum of lengths of curves in $(\mathbf{G}_m, \mathbf{d})$ from γ to σ. Since these lengths are greater than or equal to $\mathbf{d}(\gamma, \sigma)$, we have $\mathbf{d} \leqslant \mathbf{d}^*$. By Theorem 2, the endpoint map is a local isometry from $(\mathbf{G}_m, \mathbf{d})$ onto M. It follows by the definition of $\mathbf{d}^*$ that every element of $\mathbf{G}_m$ has a neighborhood on which $\mathbf{d}$ and $\mathbf{d}^*$ coincide. It only remains to verify that $\mathbf{d}^*$ is complete; but since $\mathbf{d}$ is complete and $\mathbf{d} \leqslant \mathbf{d}^*$, any $\mathbf{d}^*$-Cauchy sequence converges in $\mathbf{d}$ and hence in $\mathbf{d}^*$. □

4. Cohn-Vossen's theorem and spaces without conjugate points

We have seen that in complete locally convex spaces, the endpoint map on $(\mathbf{G}_m, \mathbf{d})$, which we may denote by $\exp_m$, is a covering map. Such an argument will be more difficult to make if we merely assume that our spaces have no conjugate points; in fact, we have only been successful under the additional assumption of local compactness. Recall that the Hopf-Rinow theorem is used to prove the corresponding theorem in Riemannian geometry. To follow this lead would require a very general version of the Hopf-Rinow theorem, and one which does not hinge on the infinite extendibility of geodesics. It turns out that, in locally compact spaces, one may substitute for the notion of infinite extendibility, that of extendibility to a closed interval. This version is essentially due to Cohn-Vossen [C-V]; also see [Bu3, p. 4]. (In these references, condition (i) below is not discussed explicitly, but the proof suffices for the theorem as stated here.)