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It follows that

$$(5.10)\qquad I(V_m, a; V_n, b) = \sum_{j=0}^{\min\{m,n\}} \prod_{i=0}^{j-1} \frac{a-b+\frac{1}{2}(m+n)-i}{a-b-\frac{1}{2}(m+n)+i} P_j .$$

We summarize our results in the following theorem.

THEOREM 5.11. *The R-matrix associated to the representation*

$$V = V_{m_1}(a_1) \otimes \cdots \otimes V_{m_k}(a_k)$$

of Y *is given by*

$$R(a-b) = \Big(\prod_{i,j=1}^{k} I(V_{m_i}, a + a_i; V_{m_j}, b + a_j)\Big)\sigma ,$$

where the intertwining operators are given by equation (5.10) and σ *is the switch map. The order of the factors in the product is such that the* (i, j)*-term appears to the left of the* (i', j')*-term iff*

$$i > i' \quad or \quad i = i' \quad and \quad j < j' .$$

6. CONCLUDING REMARKS

Since we have discussed only the Yangian associated to $\mathfrak{sl}_2$ in this paper, it may be worth-while to indicate the extent to which the results above can be generalized to the Yangian $Y(\mathfrak{a})$ associated to an arbitrary finite-dimensional complex simple Lie algebra $\mathfrak{a}$.

The definition of $Y(\mathfrak{a})$ is precisely as in (1.1), except of course that $\{I_\lambda\}$ should be an orthonormal basis of $\mathfrak{a}$ with respect to some invariant inner product. The formulae

$$\tau_a(x) = x , \qquad \tau_a(J(x)) = J(x) + ax ,$$

for $x \in \mathfrak{a}$, again define a one-parameter group of Hopf algebra automorphisms of $Y(\mathfrak{a})$, and the relation, discussed in section 5, between solutions of the quantum Yang-Baxter equation and intertwining operators between tensor products of representations of $Y(\mathfrak{a})$, which follows from the existence of the τ_a, is also valid in the general case.

The major complication which arises in the general case is that, except when $\mathfrak{a} = \mathfrak{sl}_2$, the assignment.

$$x \to x\,, \qquad J(x) \to 0$$

does not extend to a homomorphism of algebras

$$Y(\mathfrak{a}) \to U(\mathfrak{a})\,.$$

However, if $\mathfrak{a} = \mathfrak{sl}_n$, there is a substitute, namely

$$\varepsilon(x) = x\,, \qquad \varepsilon(J(x)) = \frac{1}{4}\sum_{\lambda,\mu} \text{trace}\,(x(I_\lambda I_\mu + I_\mu I_\lambda))I_\lambda I_\mu$$

(see [4]). One can now define evaluation representations of Y by pulling back representations of $\mathfrak{sl}_n$ using the homomorphisms $\varepsilon \circ \tau_a$. We make the following conjecture, generalizing the case $n = 2$ proved above:

CONJECTURE. *Every finite-dimensional irreducible representation of $Y(\mathfrak{sl}_n)$ is isomorphic to a tensor product of evaluation representations.*

If $\mathfrak{a} \neq \mathfrak{sl}_n$ for any n, it turns out that there is no homomorphism of algebras $\varepsilon\colon Y(\mathfrak{a}) \to U(\mathfrak{a})$ such that $\varepsilon(x) = x$ for all $x \in U(\mathfrak{a})$ (see [4]), and there is no obvious analogue of the evaluation representations. However, there is a straightforward generalization of Theorem 2.4 classifying the finite-dimensional irreducible representations of $Y(\mathfrak{a})$ in terms of highest weights. Such representations are now in one-to-one correspondence with l-tuples of monic polynomials $\{P_i\}$, where $l = rank\,\mathfrak{a}$. It is natural to define a fundamental representation $V_{i,a}$ of $Y(\mathfrak{a})$ to be one corresponding to a set of polynomials of the form

$$P_j(u) = 1 \quad if \quad j \neq i\,,$$
$$P_i(u) = u - a$$

for some $a \in \mathbf{C}$. As in the case of semisimple Lie algebras, it can be shown that every finite-dimensional irreducible representation of $Y(\mathfrak{a})$ (for any $\mathfrak{a}$) is a subquotient of a tensor product of fundamental representations (see [3]), and understanding the structure of the fundamental representations $V_{i,a}$ is thus an important first step towards understanding the most general representation. The case where the i^{th} node of the Dynkin diagram is extremal (i.e. joined to only one other node), and also the case where the i^{th} fundamental representation of $\mathfrak{a}$ is the adjoint representation, were discussed in [4], and the general case is dealt with in [3], except for a few nodes of E_7 and E_8 (some cases have also appeared without proof in the physics literature [10]).

As we have already mentioned, to any finite-dimensional irreducible representation of any Yangian $Y(\mathfrak{a})$ is associated a rational solution of the QYBE. Many examples of such R-matrices have been computed in the literature (see [10] for a recent summary), although these calculations are mainly restricted to the case of those fundamental representations which are irreducible as representations of $\mathfrak{a}$ (the only exception seems to be [11], which considers the case where i corresponds to the spin representation(s) of $\mathfrak{a} = \mathfrak{so}_n$). In [3] we carry out the computations for most of the remaining fundamental representations, and also for the adjoint representation of $Y(\mathfrak{sl}_n)$. However, the computation of the R-matrix associated to an arbitrary representation of $Y(\mathfrak{a})$ remains a difficult open problem.