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5. *R*-MATRICES AND INTERTWINING OPERATORS

In this section we shall prove that, after a trivial twisting, the intertwining operators between certain representations of Yangians provide rational solutions of the quantum Yang-Baxter equation. Recall that, if V is any representation of $Y = Y(\mathfrak{sl}_2)$, then, for any $a \in \mathbb{C}$, we denote by $V(a)$ its pull-back by the automorphism τ_a of Y defined in Proposition 2.5.

PROPOSITION 5.1. *Let V, W be irreducible finite-dimensional representations of Y with highest weight vectors Ω_V, Ω_W and let $a, b \in \mathbb{C}$. Then:*
 (a) *the tensor products $V(a) \otimes W(b)$ and $W(b) \otimes V(a)$ are irreducible and isomorphic except for a finite set of values $S(V, W)$ of $a - b$;*
 (b) *the unique intertwining operator*

$$I(V, a; W, b): W(b) \otimes V(a) \rightarrow V(a) \otimes W(b)$$

which maps $\Omega_W \otimes \Omega_V$ to $\Omega_V \otimes \Omega_W$ is a rational function of $a - b$ with values in $\text{Hom}(W \otimes V, V \otimes W)$.

Proof. Part (a) follows immediately from Proposition 4.2 and Corollary 4.7. For part (b), we need the following lemma.

LEMMA 5.2. *Let V, W be representations of Y and let $a \in \mathbb{C}$.*
 (a) *If V is irreducible, so is $V(a)$.*
 (b) *If $I: V \rightarrow W$ is an isomorphism of representations of Y , so is $I: V(a) \rightarrow W(a)$.*

Proof of lemma. Part (a) follows from the definition of $V(a)$. For part (b), we must show that I commutes with the action of x and $J(x)$ on $V(a)$ and $W(a)$, for all $x \in \mathfrak{sl}_2$. But this is clear, since the action of x is the same as that on V and W , and that of $J(x)$ is the same as that of $J(x) + ax$ on V and W .

Returning to the proof of Proposition 5.1, it follows from the lemma that $I(V, a; W, b)$ is a function of $a - b$, so it suffices to consider the case $b = 0$. For any $a \in \mathbb{C}$ which does not belong to the finite set $S(V, W)$, there is a unique isomorphism

$$I(V, a; W, 0) \equiv I(a): W \otimes V(a) \rightarrow V(a) \otimes W$$

of representations of Y such that

$$(5.3) \quad I(a) (\Omega_W \otimes \Omega_V) = \Omega_V \otimes \Omega_W.$$

Choose bases of $V \otimes W$ and $W \otimes V$ and let $\{I_\lambda\}$ be a basis of \mathfrak{sl}_2 ; write $I(a)$ also for the matrix of $I(a)$ with respect to these bases. Let A_λ, B_λ be the matrices of I_λ and $J(I_\lambda)$ acting on $W \otimes V(a)$; and let A'_λ and B'_λ refer similarly to $V(a) \otimes W$. Then, $I(a)$ commutes with the action of Y if and only if $I(a)$ satisfies the following system of homogeneous linear equations:

$$A_\lambda I(a) = I(a) A'_\lambda, \quad B_\lambda I(a) = I(a) B'_\lambda, \quad \text{for all } \lambda.$$

We know that, if $a \notin S(V, W)$, these equations have a unique solution satisfying equation (5.3). By elementary linear algebra, the solution is a rational function of the entries of the matrices $A_\lambda, A'_\lambda, B_\lambda, B'_\lambda$. Since A_λ, A'_λ are independent of a and B_λ, B'_λ are linear in a , the result follows.

Definition 5.4. Let V be a finite-dimensional irreducible representation of Y . Then, the R -matrix associated to V is the function $R(a-b)$ with values in $\text{End}(V \otimes V)$ given by

$$R(a-b) = I(V, a; V, b) \sigma,$$

where $\sigma \in \text{End}(V \otimes V)$ is the switch of the two factors.

THEOREM 5.5. *Let V be a finite-dimensional irreducible representation of Y . Then the R -matrix associated to V is a rational solution of the quantum Yang-Baxter equation:*

$$(5.6) \quad R^{12}(a-b)R^{13}(a-c)R^{23}(b-c) = R^{23}(b-c)R^{13}(a-c)R^{12}(a-b).$$

Proof. We note first some simple commutation relations between the intertwining operator $I(a-b) \equiv I(V, a; V, b)$ and the switch map σ . For example, we have

$$\sigma^{12} I^{13}(a-c) \sigma^{12} = I^{23}(a-c).$$

by an easy computation. Similarly,

$$\sigma^{12} \sigma^{13} I^{23}(b-c) \sigma^{13} \sigma^{12} = I^{12}(b-c).$$

Hence,

$$\begin{aligned} R^{12}(a-b)R^{13}(a-c)R^{23}(b-c) &= I^{12}(a-b)\sigma^{12}I^{13}(a-c)\sigma^{13}I^{23}(b-c)\sigma^{23} \\ &= I^{12}(a-b)I^{23}(a-c)\sigma^{12}\sigma^{13}I^{23}(b-c)\sigma^{23} \\ &= I^{12}(a-b)I^{23}(a-c)I^{12}(b-c)\sigma^{12}\sigma^{13}\sigma^{23}. \end{aligned}$$

Similarly,

$$R^{23}(b-c)R^{13}(a-c)R^{12}(a-b) = I^{23}(b-c)I^{12}(a-c)I^{23}(a-b)\sigma^{23}\sigma^{13}\sigma^{12}.$$

Hence, in view of the relation

$$\sigma^{12} \sigma^{13} \sigma^{23} = \sigma^{23} \sigma^{13} \sigma^{12}$$

in the symmetric group on three letters, the equation to be proved is

$$(5.7) \quad I^{12}(a-b)I^{23}(a-c)I^{12}(b-c) = I^{23}(b-c)I^{12}(a-c)I^{23}(a-b) .$$

Note that both sides of equation (5.7) define intertwining operators

$$V(c) \otimes V(b) \otimes V(a) \rightarrow V(a) \otimes V(b) \otimes V(c)$$

which fix the tensor product of the highest weight vectors in V . Hence, regarded as functions on \mathbf{C}^3 with values in $\text{End}(V \otimes V \otimes V)$, they agree on the complement of the set S of $(a, b, c) \in \mathbf{C}^3$ where $V(c) \otimes V(b) \otimes V(a)$ or $V(a) \otimes V(b) \otimes V(c)$ is reducible. It follows from part (a) of Proposition 5.1 that S intersects each complex line parallel to one of the axes in \mathbf{C}^3 in at most finitely many points. It is easy to see that the complement of such a set is Zariski dense in \mathbf{C}^3 . Since the two sides of equation (5.7) are rational functions which agree on a Zariski dense set, they are equal.

Remark. We have used the following simple fact about intertwining operators. Let U, V and W be representations of a Yangian $Y(\mathfrak{sl}_2)$ and let $I: U \otimes V \rightarrow V \otimes U$ be an intertwining operator. Then

$$I^{12}: U \otimes V \otimes W \rightarrow V \otimes U \otimes W$$

and

$$I^{23}: W \otimes U \otimes V \rightarrow W \otimes V \otimes U$$

are intertwining operators. While this is obvious enough, it should be noted that

$$I^{13}: U \otimes W \otimes V \rightarrow V \otimes W \otimes U$$

is *not* an intertwining operator in general.

We conclude this general discussion by showing that, up to a sign change in the parameter, the R -matrix $R(u)$ we have associated to a representation of Y is the same as that constructed using the “universal R -matrix” (see Theorem 3 of [4]). Set

$$\tilde{R}(u) = R(-u) .$$

Then, by Theorem 4 of [4], it suffices to prove that

$$(5.8) \quad P_{\lambda}^{+}(a, b) \tilde{R}(b-a) = \tilde{R}(b-a) P_{\lambda}^{-}(a, b)$$

where

$$P_{\lambda}^{\pm}(a, b) = (\rho \otimes \rho) ((J(I_{\lambda}) + aI_{\lambda}) \otimes 1 + 1 \otimes (J(I_{\lambda}) + bI_{\lambda}) + \frac{1}{2} [I_{\lambda} \otimes 1, \Omega]) ,$$

$\rho: Y \rightarrow \text{End}(V)$ is the action of Y on V and $\{I_{\lambda}\}$ is an orthonormal basis of \mathfrak{sl}_2 . In terms of intertwining operators, equation (5.8) asserts that

$$P_{\lambda}^{+}(a, b)I(a-b) = I(a-b)\sigma P_{\lambda}^{-}(a, b)\sigma .$$

But it is easy to see that

$$\sigma P_{\lambda}^{-}(a, b)\sigma = P_{\lambda}^{+}(b, a) .$$

Hence, we must prove that

$$P_{\lambda}^{+}(a, b)I(a-b) = I(a-b)P_{\lambda}^{+}(b, a) .$$

But this is simply the statement that

$$I(a-b): V(b) \otimes V(a) \rightarrow V(a) \otimes V(b)$$

commutes with the action of $J(I_{\lambda})$.

We shall now apply these results to compute the R -matrices associated to every finite-dimensional irreducible representation of Y . By Theorem 4.11, every such representation is of the form

$$V = V_{m_1}(a_1) \otimes \cdots \otimes V_{m_k}(a_k).$$

The intertwining operator

$$I(a-b): V(b) \otimes V(a) \rightarrow V(a) \otimes V(b)$$

can be computed as the product of k^2 intertwining operators of the form $I(V_m, a; V_n, b)$, each of which effects an interchange of nearest neighbours. Since such an operator commutes, in particular, with the action of \mathfrak{sl}_2 , it can be written in the form

$$(5.9) \quad I(V_m, a; V_n, b) = \sum_{j=0}^{\min\{m, n\}} c_j P_{m+n-2j} ,$$

where

$$P_{m+n-2j}: V_n \otimes V_m \rightarrow V_m \otimes V_n$$

is the projection onto the irreducible component of

$$V_m \otimes V_n \cong \bigotimes_{j=0}^{\min\{m, n\}} V_{m+n-2j}$$

of type V_{m+n-2j} . We have $c_0 = 1$ since $I(V_m, a; V_n, b)$ preserves the tensor products of the highest weight vectors.

To compute $I(V_m, a; V_n, b)$, let $\Omega_j, j = 0, 1, \dots, \min\{m, n\}$, be a highest weight vector in $V_n \otimes V_m$ of weight $m + n - 2j$; then, the vector Ω'_j obtained by switching the order of the factors in Ω_j is a highest weight vector in $V_m \otimes V_n$ of the same weight, and we have

$$I(V_m, a; V_n, b) (\Omega_j) = \Omega'_j.$$

Further, it is easy to see that, for $j > 0$, $(x^+ \otimes 1) \cdot \Omega_j$ is an \mathfrak{sl}_2 -highest weight vector of weight $m + n - 2j + 2$; it is non-zero, since otherwise Ω_j would be annihilated by $x^+ \otimes 1$ and by $1 \otimes x^+$, contradicting the assumption $j > 0$. Hence, we may assume that

$$(x^+ \otimes 1) \cdot \Omega_j = \Omega_{j-1}$$

for $j > 0$. Switching the order of the factors, we have

$$(x^+ \otimes 1) \cdot \Omega'_j = -\Omega'_{j-1}.$$

By Proposition 4.2 (and its proof), Ω_j is a Y -highest weight vector in $V_n(b) \otimes V_m(a)$ if

$$b - a = \frac{1}{2} (m + n) - j + 1.$$

It follows from the formula for the co-multiplication in Definition 1.1 that, in the representation $V_n(b) \otimes V_m(a)$,

$$J(x^+) \cdot \Omega_j = \left(b - a - \frac{1}{2} (m + n) + j - 1 \right) (x^+ \otimes 1) \cdot \Omega_j,$$

and that in the representation $V_m(a) \otimes V_n(b)$,

$$J(x^+) \cdot \Omega'_j = \left(a - b - \frac{1}{2} (m + n) + j - 1 \right) (x^+ \otimes 1) \cdot \Omega'_j.$$

The equation

$$I(V_m, a; V_n, b) (J(x^+) \cdot \Omega_j) = J(x^+) \cdot (I(V_m, a; V_n, b) \Omega_j)$$

now gives

$$\frac{c_j}{c_{j-1}} = \frac{a - b + \frac{1}{2} (m + n) - j + 1}{a - b - \frac{1}{2} (m + n) - j + 1}.$$

It follows that

$$(5.10) \quad I(V_m, a; V_n, b) = \sum_{j=0}^{\min\{m, n\}} \prod_{i=0}^{j-1} \frac{a - b + \frac{1}{2}(m+n) - i}{a - b - \frac{1}{2}(m+n) + i} P_j.$$

We summarize our results in the following theorem.

THEOREM 5.11. *The R -matrix associated to the representation*

$$V = V_{m_1}(a_1) \otimes \cdots \otimes V_{m_k}(a_k)$$

of Y is given by

$$R(a-b) = \left(\prod_{i,j=1}^k I(V_{m_i}, a + a_i; V_{m_j}, b + a_j) \right) \sigma,$$

where the intertwining operators are given by equation (5.10) and σ is the switch map. The order of the factors in the product is such that the (i, j) -term appears to the left of the (i', j') -term iff

$$i > i' \quad \text{or} \quad i = i' \quad \text{and} \quad j < j'.$$

6. CONCLUDING REMARKS

Since we have discussed only the Yangian associated to \mathfrak{sl}_2 in this paper, it may be worth-while to indicate the extent to which the results above can be generalized to the Yangian $Y(\mathfrak{a})$ associated to an arbitrary finite-dimensional complex simple Lie algebra \mathfrak{a} .

The definition of $Y(\mathfrak{a})$ is precisely as in (1.1), except of course that $\{I_\lambda\}$ should be an orthonormal basis of \mathfrak{a} with respect to some invariant inner product. The formulae

$$\tau_a(x) = x, \quad \tau_a(J(x)) = J(x) + ax,$$

for $x \in \mathfrak{a}$, again define a one-parameter group of Hopf algebra automorphisms of $Y(\mathfrak{a})$, and the relation, discussed in section 5, between solutions of the quantum Yang-Baxter equation and intertwining operators between tensor products of representations of $Y(\mathfrak{a})$, which follows from the existence of the τ_a , is also valid in the general case.