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AN ELEMENTARY PROOF OF A THEOREM ON QUADRATIC FORMS OVER THE RATIONAL NUMBERS

by David B. LEEP

INTRODUCTION

It is well known, and easy to prove, that each positive rational number is a sum of four rational squares. The main idea of the proof is that the set of nonzero rational numbers which are sums of four squares forms a group under multiplication.

Let F be a field of characteristic $\neq 2$, $F^{\times}$ the nonzero elements of F, and let $\langle a_1, \ldots, a_n \rangle$ denote the quadratic form $a_1 x_1^2 + \ldots + a_n x_n^2$ where $a_i \in F^{\times}$. Let

$$D_F(\langle a_1, \ldots, a_n \rangle) = \{c \in F^{\times} \mid c = \sum_{i=1}^{n} a_i x_i^2, x_i \in F\}.$$

Then $D_F(\langle 1, a, b, ab \rangle)$ is a multiplicative subgroup of $F^{\times}$. (See Lemma 1.3.) Let $\mathbf{Q}$ be the field of rational numbers. The goal of this paper is to give a new and elementary proof of the following theorem of which the result above is a special case.

MAIN THEOREM. *Let* $a, b \in \mathbf{Q}^{\times}$. *Then*

$$D_{\mathbf{Q}}(\langle 1, a, b, ab \rangle) = \begin{cases} \mathbf{Q}^{\times}_{>0} \text{ (positive rationals)} & \text{if } a, b > 0 \\ \mathbf{Q}^{\times} & \text{otherwise}. \end{cases}$$

There are essentially three ways to prove the Main Theorem at present. One way is to use the Hasse-Minkowski theorem ([La], p. 168). This is, however, a difficult theorem to prove. Proofs of the Hasse-Minkowski theorem rely on Dirichlet's theorem on primes in an arithmetic progression ([Se], [BS]), class field theory ([Om]), or Gauss' theory on the existence of certain types of binary quadratic forms ([Ca]). (Actually, Skolem showed in [Sk] that a weaker analytic result than Dirichlet's theorem suffices to give a proof of the Hasse-Minkowski theorem.) A second way to prove the Main Theorem is to use Meyer's theorem that an indefinite 5-dimensional quadratic form over $\mathbf{Q}$ is isotropic. This theorem was originally proved using the classical integral theory of quadratic forms over the integers and also depends on Dirichlet's theorem or Gauss' theory mentioned above. A third way is to use the so called weak Hasse-Minkowski theorem. A proof of this can be found in [La], p. 174-178, but knowledge is required of Witt rings, local fields, exact sequences, and Springer's theory for quadratic forms over local fields.

Until now, no proof of the Main Theorem, much less an elementary one, has appeared exploiting the fact that $D_{\mathbf{Q}}(\langle 1, a, b, ab\rangle)$ is a multiplicative subgroup of $\mathbf{Q}^{\times}$. We present a truly elementary proof below using nothing more exotic than the notion of quadratic residues and the Möbius function.

We follow basic terminology and notation as found in [La]. In particular, a quadratic form $\langle a_1, \ldots, a_n\rangle$ is isotropic over F if there exist $x_1, \ldots, x_n \in F$, not all zero, such that $\sum_{i=1}^{n} a_i x_i^2 = 0$. We have the orthogonal sum $\langle a_1, \ldots, a_m\rangle \perp \langle b_1, \ldots, b_n\rangle = \langle a_1, \ldots, a_m, b_1, \ldots, b_n\rangle$ and $\langle\langle a, b\rangle\rangle$ stands for $\langle 1, a, b, ab\rangle$.

I wish to thank T.Y. Lam for the proof of Proposition 1.4 which is much simpler than my original proof.

§ 1. Reductions to prove the Main Theorem

1.1. Main Theorem. *Let* $a, b \in \mathbf{Q}^{\times}$. *Then*

$$D_{\mathbf{Q}}(\langle 1, a, b, ab\rangle) = \begin{cases} \mathbf{Q}^{\times}_{>0} & \text{if } a, b > 0 \\ \mathbf{Q}^{\times} & \text{otherwise}. \end{cases}$$

We begin by stating some basic results needed to prove Theorem 1.1.

1.2. Lemma. *Let* $q = \langle a_1, \ldots, a_n\rangle$, $a_i \in F^{\times}$.

(a) *If* q *is isotropic over* F, *then* $D_F(q) = F^{\times}$.

(b) *Let* $c \in F^{\times}$. *Then* $q \perp \langle c\rangle$ *is isotropic over* F *if and only if* $-c \in D_F(q)$.

Proof. (a) Let $c \in F^{\times}$ be given. An appropriate linear change of variable lets us assume $q(1, 0, \ldots, 0) = 0$. Then

$$q(x_1, \ldots, x_n) = x_1\left(\sum_{i=2}^{n} b_i x_i\right) + Q(x_2, \ldots, x_n)$$

where some $b_i \neq 0$. Choose $a_2, \ldots, a_n \in F$ such that $\sum_{i=2}^{n} b_i a_i \neq 0$ and let