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CAUCHY RESIDUES AND DE RHAM HOMOLOGY

by Birger IVERSEN

This paper represents my third attempt to write up a suitable generalization of the classical Cauchy Residue theorem. As I pushed the earlier versions for naturality and generality I was ultimately lead to a new foundation of de Rham homology free of the theory of distributions but relying on basic sheaf theory much like the Borel Moore homology theory.

Singular homology and de Rham homology agree on a smooth manifold. The whole point in introducing de Rham homology is the posibility of alternative representations of homology classes. This is amply illustrated by the general Cauchy residue formula given at the end of the paper.

I would like to thank the Mittag Leffler Institute at Stockholm for hospitality while this paper was worked out.

1. COMPACT CHAINS

Let X denote a smooth n-dimensional manifold. For an integer p we let $\Gamma(X, \Omega^p)$ denote the space of $\mathbf{C}$-valued differential p-forms on X. By a *compact p-chain* on X we understand a $\mathbf{C}$-valued linear form T on $\Gamma(X, \Omega^p)$ for which there exists a compact subset K of X such that

$$(1.1) \qquad < T, \omega > \; = 0\,, \quad \omega \in \Gamma(X, \Omega^p)\,, \quad \mathrm{Supp}\,(\omega) \cap K = \emptyset\,,$$

where the bracket denotes simple evaluation of a linear form. The compact p-chain T on X gives rise to a $(p-1)$-chain bT on X given by

$$(1.2) \qquad < bT, \omega > \; = \; < T, d\omega > \,, \quad \omega \in \Gamma(X, \Omega^{p-1})\,.$$

The operator b makes the compact p-chains on X into a complex, which we denote $D^c_\bullet(X, \mathbf{C})$. A compact p-chain T is *closed* if $bT = 0$. By a compact *p-cycle* we understand a closed p-chain, while a compact *p-boundary* is a p-cycle of the form bW, where W is a compact $(p+1)$-chain. We say that p-cycles S and T are *homologous* if $T - S$ is a p-boundary. An explicit relation of the form

$$T - S = bW, \quad W \in D^c_{p+1}(X, \mathbf{C}), \tag{1.3}$$

is called a *de Rham homology* from S to T. Explicit de Rham homologies are often constructed on the basis of Stokes theorem, compare the formulas to the right of the drawings in section 7.

Let us make the important observation that homologous p-chains have the same evolution on any closed p-form. The group of de Rham homology classes is denoted by

$$H^c_p(X, \mathbf{C}) = H_p D^c_\bullet(X, \mathbf{C}). \tag{1.4}$$

The letter c in the homology symbol is borrowed from Haefliger's exposé in [1].

A smooth map $f: X \to Y$ will induce a chain map from the complex of compact chains on X to the complex of compact chains on Y

$$f_*: D^c_\bullet(X, \mathbf{C}) \to D^c_\bullet(Y, \mathbf{C}).$$

To see this notice that a given compact subset K on X and a p-form ω on Y supported outside $f(K)$ pulls back to a form $f^*\omega$ supported outside K. We can now define $f_* T$ by the formula

$$<f_* T, \omega> = <T, f^*\omega>, \quad T \in D^c_p(X, \mathbf{C}), \omega \in \Gamma(Y, \Omega^p). \tag{1.5}$$

These remarks make it clear how to turn de Rham homology into a covariant functor on the smooth category.

Zero cycles. Evaluation of a compact zero cycle Z against the constant function 1 defines the *degree of the zero cycle*

$$\mathrm{dg}(Z) = <Z, 1>, \quad Z \in D^c_0(X, \mathbf{C}). \tag{1.6}$$

A point $x \in X$ defines a zero cycle of degree 1, the *Dirac 0-cycle* δ_x given by

$$<\delta_x, \varphi> = \varphi(x), \quad \varphi \in \Gamma(X, \Omega^0). \tag{1.7}$$

A continuously differentiable curve $\gamma: [a, b] \to X$ with endpoints $x = \gamma(a)$ and $y = \gamma(b)$ defines a de Rham homology from δ_x to δ_y:

$$\int_\gamma d\varphi = \varphi(y) - \varphi(x), \quad \varphi \in \Gamma(X, \Omega^0). \tag{1.8}$$

In case X is connected, then all zero cycles of degree zero are homologous to zero as it follows from the result of the next section.

A smooth map $f: X \to Y$ will preserve the degree of a zero cycle in the sense that

(1.9) $$\operatorname{dg}(f_* T) = \operatorname{dg} T\,, \quad T \in D_0^c(X, \mathbf{C})\,,$$

as it follows from (1.5).

The reader is invited to replace $\mathbf{C}$ by $\mathbf{R}$ and change the meaning of the symbol Ω from complex to real differential forms.

2. Biduality

In this section we shall show that de Rham cohomology can be calculated as the linear dual of de Rham homology in the same way singular cohomology can be obtained from singular homology.

(2.1) Theorem. *Let X denote a smooth manifold. Evaluation of a compact p-chain against a p-form induces an isomorphism*

$$H^p(X, \mathbf{C}) \xrightarrow{\sim} \operatorname{Hom}(H_p^c(X, \mathbf{C}), \mathbf{C})$$

for all integers p.

Proof. The hart of the matter is of sheaf theoretic nature, so we start with a brief review during which the reader is invited to change the meaning of the letter X to denote a general locally compact space and the letter $\mathbf{C}$ to denote an arbitrary field. For notation and details the reader may consult [5] V.1, and the references given there.

To a *soft* $\mathbf{C}$-sheaf F on X we can associate the sheaf $F^\vee$ whose sections over the open subset U of X are given by

(2.2) $$\Gamma(U, F^\vee) = \operatorname{Hom}(\Gamma_c(U, F), \mathbf{C})$$

Restriction in the sheaf $F^\vee$ from U to a smaller open subset V is the $\mathbf{C}$-linear dual of "extension by zero"

$$\Gamma_c(V, F) \to \Gamma_c(U, F)\,, \quad V \subseteq U\,.$$

The presheaf $F^\vee$ we have described is actually a sheaf and indeed a *soft* sheaf. This allows us to iterate the construction and form $F^{\vee\vee}$. We shall construct a natural *biduality morphism* of $\mathbf{C}$-sheaves on X

(2.3) $$F \to F^{\vee\vee}\,.$$

To this end consider the tautological evaluation

$$\Gamma(U, F^\vee) \times \Gamma_c(U, F) \to \mathbf{C}.$$

This can be modified to yield a pairing

$$\mathrm{ev}: \Gamma_c(U, F^\vee) \times \Gamma(U, F) \to \mathbf{C}$$

namely, for $T \in \Gamma_c(U, F^\vee)$ and $\omega \in \Gamma(U, F)$ choose $\nu \in \Gamma_c(U, F)$, such that ω and ν has the same restriction to $\mathrm{Supp}(T)$, and put $\mathrm{ev}(T, \omega) = T(\nu)$. The evaluation map may be interpreted as a transformation

$$a_U: \Gamma_c(U, F^\vee) \to \mathrm{Hom}(\Gamma(U, F), \mathbf{C}). \tag{2.4}$$

An open subset V of U gives rise to a commutative diagram

$$\begin{array}{ccc} \Gamma_c(V, F^\vee) & \xrightarrow{a_V} & \mathrm{Hom}(\Gamma(V, F), \mathbf{C}) \\ \downarrow & & \downarrow \\ \Gamma_c(U, F^\vee) & \xrightarrow{a_U} & \mathrm{Hom}(\Gamma(U, F), \mathbf{C}) \end{array} \tag{2.5}$$

where the first vertical arrow is "extension by zero" in the soft sheaf $F^\vee$ and the second vertical arrow is the linear dual of restriction in the sheaf F. Let us return to the open subset U and consider the composite

$$\Gamma(U, F) \to \mathrm{Hom}(\mathrm{Hom}(\Gamma(U, F), \mathbf{C}), \mathbf{C}) \xrightarrow{a_U^*} \Gamma(U, F^{\vee\vee})$$

where the first arrow is the biduality map from linear algebra. By variation of U we obtain the biduality morphism $b: F \to F^{\vee\vee}$ announced in (2.3).

Let us now return to the situation at hand and consider the biduality morphism for the de Rham complex.

$$b: \Omega^\bullet \to \Omega^{\bullet\vee\vee} \tag{2.6}$$

which we shall prove to be a quasi-isomorphism. The question is local, so it suffices to check the case $X = \mathbf{R}^n$, which can be done by the Poincaré Lemma with and without compact support. Both complexes are made of soft sheaves, so we lean on the fact, implicit in the definition of a manifold, that X is countable at infinity to conclude that b induces isomorphisms, compare [5] IV.2.2,

$$H^p(X, \mathbf{C}) \xrightarrow{\sim} H^p\Gamma(X, \Omega^{\bullet\vee\vee}), \quad p \in \mathbf{N}.$$

In order to identify the right hand side notice first that

$$\Gamma(X, \Omega^{\bullet\vee\vee}) = \mathrm{Hom}(\Gamma_c(X, \Omega^{\bullet\vee}), \mathbf{C})$$

and second, that the map a_X introduced in (2.4) induces an isomorphism

$$a : \Gamma_c(X, \Omega^{\cdot\vee}) \xrightarrow{\sim} D^c_{\cdot}(X, \mathbf{C}) . \tag{2.7}$$

Collect this together to conclude the proof. Q.E.D.

The de Rham homology as defined here agrees with the original theory based on currents [6]: the inclusion of the complex of currents in $\Omega.^{\vee}$ is a quasi-isomorphism as can be seen by the method used in the last third of the proof above.

As a consequence of the isomorphism (2.7) we can of course redefine de Rham homology as

$$H^c_p(X, \mathbf{C}) = H_p \Gamma_c(X, \Omega^{\cdot\vee}) . \tag{2.8}$$

If the letter c is dropped we obtain Borel Moore homology, compare [5] IX and the references given there.

The biduality theorem 2.1 is certainly related to that of Verdier [7], [1]. In fact most of the material presented here may be extended to a context of similar generality. I hope to return to this point in the near future.

3. Smooth singular homology

Let us consider an n-dimensional smooth manifold X. Integration over smooth singular simplexes defines a map

$$S^{\infty}_{\cdot}(X, \mathbf{C}) \to D^c_{\cdot}(X, \mathbf{C}) \tag{3.1}$$

from the complex of smooth singular simplexes to the complex of compact chains on X.

(3.2) THEOREM. *Integration induces an isomorphism*

$$H^{\infty}_{\cdot}(X, \mathbf{C}) \xrightarrow{\sim} H^c_{\cdot}(X, \mathbf{C})$$

from smooth singular homology to de Rham homology.

Proof. Let us first discuss *Mayer-Vietoris* sequences in de Rham homology. For open subsets U and V of X a Mayer-Vietoris sequence arises from the following exact sequence of complexes

$$0 \to \Gamma_c(U \cap V, \Omega^{\cdot\vee}) \xrightarrow{+} \Gamma_c(U, \Omega^{\cdot\vee}) \oplus \Gamma_c(V, \Omega^{\cdot\vee}) \xrightarrow{-} \Gamma_c(U \cup V, \Omega^{\cdot\vee}) \to 0 . \tag{3.3}$$

The main reason for the exactness of the sequence is

$$H_c^1(U \cap V, \Omega^{q\vee}) = 0\,, \quad q \in \mathbf{N}\,,$$

compare [5] III. 7.5. The vanishing of the cohomology group follows from the fact that $\Omega^{q\vee}$ is a flabby sheaf.

In singular homology the Mayer-Vietoris sequence originates from the tautological exact sequence

$$0 \to S_{\bullet}^{\infty}(U \cap V, \mathbf{C}) \to S_{\bullet}^{\infty}(U, \mathbf{C}) \oplus S_{\bullet}^{\infty}(V, \mathbf{C}) \to S_{\bullet}^{\infty}(U, V; \mathbf{C}) \to 0\,,$$

where $S_{\bullet}^{\infty}(U, V; \mathbf{C})$ is the complex based on singular simplexes supported entirely by U or entirely by V. The difficult part is to prove that the inclusion

$$S_{\bullet}^{\infty}(U, V, \mathbf{C}) \to S_{\bullet}^{\infty}(U \cup V, \mathbf{C})$$

is a quasi-isomorphism, compare [8].

This description makes it obvious that the Mayer-Vietoris sequences in the two theories are connected by a commutative ladder.

A second common feature of the two theories is that given a manifold X which is the disjoint union of the family (X_α) of open subsets, then

(3.4) $$\bigoplus_{\alpha} H_{\bullet}(X_\alpha, \mathbf{C}) \xrightarrow{\sim} H_{\bullet}(X, \mathbf{C})$$

as it follows from the Borel-Heine theorem.

Let us now investigate the case $X = \mathbf{R}^n$. The homology groups are for both theories, compare (2.1)

$$H_0(\mathbf{R}^n, \mathbf{C}) = \mathbf{C}\,, \quad H_i(\mathbf{R}^n, \mathbf{C}) = 0\,, \quad i \geqslant 1\,.$$

The transition from $H_{\bullet}^{\infty}(\mathbf{R}^n, \mathbf{C})$ to $H_{\bullet}^{c}(\mathbf{R}^n, \mathbf{C})$ is an isomorphism as it follows from the discussion of zero cycles at the end of section 1, see also (2.1).

The result follows by *Fribourg-induction*, an elementary but ingenious induction procedure based on the Mayer-Vietoris sequence, see [2] or [4] where this method is used for the proof of Poincaré duality and the Künneth theorem. The last reference attributes this method to the master thesis of C. Auderset, Fribourg 1968. Q.E.D.

Let us record that the canonical isomorphism from smooth singular homology to singular homology

(3.5) $$H_{\bullet}^{\infty}(X, \mathbf{C}) \xrightarrow{\sim} H_{\bullet}(X, \mathbf{C})$$

likewise can be established by Fribourg-induction.

If we combine theorem (3.2) and the biduality theorem (2.1) we obtain what is usually known as the

(3.6) DE RHAM THEOREM. *Integration over smooth singular simplexes induces an isomorphism*

$$H^{\bullet}(X, \mathbf{C}) \xrightarrow{\sim} H^{\bullet}_{\infty}(X, \mathbf{C})$$

from de Rham cohomology to smooth singular cohomology.

4. Relative de Rham homology

Let us start by some general remarks on the support of a compact p-chain T on a smooth n-dimensional manifold X. Since we can realize T as a section in the sheaf $\Omega^{p\vee}$ the general sheaf theoretic notion of support applies: The *support* of T, $\operatorname{Supp}(T)$ is the smallest closed subset Z of X, such that the restriction of T to $X - Z$ is zero.

(4.1) EXAMPLE. Integration over an oriented compact p-dimensional submanifold K with boundary defines a compact p-chain κ with $\operatorname{Supp}(\kappa) = K$. From Stokes formula

$$\int_K d\omega = \int_{\partial K} \omega\,, \quad \omega \in \Gamma(X, \Omega^p)\,,$$

we conclude that $\operatorname{Supp}(b\kappa) = \partial K$.

Let us now consider the inclusion $j\colon U \to X$ of an open subset U of X. The induced map

$$j_*\colon D_p^c(U, \mathbf{C}) \to D_p^c(X, \mathbf{C})\,, \quad p \in \mathbf{N}\,,$$

is injective since we may interpret j_* as "extension by zero" in the sheaf $\Omega_p^{\vee}$, compare (2.5). A compact p-chain T on X belongs to the image of j_* if and only if $\operatorname{Supp}(T) \subseteq U$. The complex $D_{\bullet}^c(X, U; \mathbf{C})$ of *relative compact chains* is defined to fit into the following exact sequence

$$0 \to D_{\bullet}^c(U, \mathbf{C}) \xrightarrow{j_*} D_{\bullet}^c(X, \mathbf{C}) \to D_{\bullet}^c(X, U; \mathbf{C}) \to 0\,. \tag{4.2}$$

On this basis we can define the *relative de Rham* homology group

$$H_p(X, U; \mathbf{C}) = H_p D_{\bullet}^c(X, U; \mathbf{C})\,, \quad p \in \mathbf{N}\,.$$

In more concrete terms we can describe this homology group as

(4.3) $$\{Z \in D_p^c(X, \mathbf{C}) \mid \operatorname{Supp}(bZ) \subseteq U\} \Big/ \begin{array}{l}\{bW \mid W \in D_{p+1}^c(X, \mathbf{C})\} \\ + \{Z \in D_p(X, \mathbf{C}) \mid \operatorname{Supp}(Z) \subseteq U\}\end{array}$$

From the exact sequence (4.2) we deduce the homology sequence

(4.4) $$\begin{array}{c}\to H_p^c(U, \mathbf{C}) \to H_p^c(X, \mathbf{C}) \to H_p^c(X, U; \mathbf{C}) \\ \to H_{p-1}^c(U, \mathbf{C}) \to H_{p-1}^c(X, \mathbf{C}) \to\end{array}$$

Let $f: X \to Y$ denote a smooth map, U an open subset of X and V an open subset of Y containing $f(U)$. Let us notice that

(4.5) $$\operatorname{Supp}(f_* T) \subseteq f(\operatorname{Supp}(T)), \quad T \in D_p^c(X, \mathbf{C}).$$

These remarks make it evident, that de Rham homology is a covariant functor on the category of pairs consisting of a manifold and one of its open subspaces.

(4.6) *Excision.* Let Z be a closed subset of X and Y an open subset of X containing Z. The inclusion of $V = Y - Z$ in $U = X - Z$ induces an isomorphism

$$H_\bullet^c(Y, V; \mathbf{C}) \xrightarrow{\sim} H_\bullet^c(X, U; \mathbf{C}).$$

Proof. Let $i: Z \to X$ denote the inclusion. From the fact that $\Omega^{\bullet\vee}$ consists of soft sheaves we deduce an exact sequence

$$0 \to \Gamma_c(U, \Omega^{\bullet\vee}) \to \Gamma_c(X, \Omega^{\bullet\vee}) \to \Gamma_c(Z, i^*\Omega^{\bullet\vee}) \to 0$$

compare [5] III. 7.6. This alows us to make the identification

(4.7) $$D_\bullet^c(X, U; \mathbf{C}) \xrightarrow{\sim} \Gamma_c(Z, i^*\Omega^{\bullet\vee}), \quad Z = X - U.$$

The expression on the right hand side is unchanged, when X is replaced by Y and U by V. Q.E.D.

(4.8) *Continuity.* Let (X_α) be an outward directed open covering of the manifold X. For any open subset U of X we have that

$$\lim_{\to\alpha} H_\bullet^c(X_\alpha, U \cap X_\alpha; \mathbf{C}) = H_\bullet^c(X, U; \mathbf{C})$$

Proof. As a consequence of the theorem of Borel-Heine, see possibly [5] III. 5.2, we find that

$$\lim_{\to} D_\bullet^c(X_\alpha, \mathbf{C}) = D_\bullet^c(X, \mathbf{C})$$

and similarly with X replaced by U and X_α replaced by $U \cap X_\alpha$. Using this and the exact sequence 4.2 we get that

$$\lim_{\rightarrow} D^c_\bullet(X_\alpha, U \cap X_\alpha; \mathbf{C}) = D^c_\bullet(X, U; \mathbf{C})$$

from which the result follows by passing to homology. Q.E.D.

Let us also notice that in case X is the disjoint union of a family (X_α) of open subsets we have that

$$\bigoplus_\alpha H^c_\bullet(X_\alpha, U \cap X_\alpha; \mathbf{C}) \xrightarrow{\sim} H^c_\bullet(X, U; \mathbf{C}). \tag{4.9}$$

5. Stokes formula

Let us consider the open subset U of the n-dimensional smooth manifold X and the resulting exact sequences

$$\begin{aligned} &\rightarrow H^c_p(X, \mathbf{C}) \rightarrow H^c_p(X, U; \mathbf{C}) \xrightarrow{b} H^c_{p-1}(U, \mathbf{C}) \xrightarrow{j_*} H^c_{p-1}(X, \mathbf{C}) \rightarrow \\ &\leftarrow H^p(X, \mathbf{C}) \leftarrow H^p(X, U; \mathbf{C}) \xleftarrow{\partial} H^{p-1}(U, \mathbf{C}) \xleftarrow{j^*} H^{p-1}(X, \mathbf{C}) \leftarrow \end{aligned} \tag{5.1}$$

where the first is discussed in the previous section and the second is the sheaf cohomology sequence. The relative term in the second sequence is often written

$$H^p_Z(X, \mathbf{C}) = H^p(X, U; \mathbf{C}), \quad Z = X - U. \tag{5.2}$$

We can now extend the biduality theorem (2.1).

(5.3) Theorem. *The cohomology sequence above is dual to the homology sequence. In particular we have a Stoke's formula*

$$<b\alpha, \omega> = <\alpha, \partial\omega>$$

for $\alpha \in H^c_p(X, U; \mathbf{C})$ *and* $\omega \in H^{p-1}(U, \mathbf{C})$.

Proof. The first sequence arises from the following short exact sequence of complexes, compare (4.2) and (4.7),

$$0 \rightarrow \Gamma_c(U, \Omega^{\bullet\vee}) \xrightarrow{j_*} \Gamma_c(X, \Omega^{\bullet\vee}) \rightarrow \Gamma_c(Z, \Omega^{\bullet\vee}) \rightarrow 0.$$

In order to calculate the second sequence we depart from the flabby resolution $\Omega^{\bullet\vee\vee}$ of $\mathbf{R}$ established in the proof of the biduality theorem (2.1).

The basic philosofy being that flabby sheaves are acyclic for local cohomology, [5] II. 9.3. Thus we can calculate the cohomology sequence (5.1) from the short exact sequence

$$0 \leftarrow \Gamma(U, \Omega^{\cdot \vee \vee}) \xleftarrow{j^*} \Gamma(X, \Omega^{\cdot \vee \vee}) \leftarrow \Gamma_Z(X, \Omega^{\cdot \vee \vee}) \leftarrow 0 .$$

According to formula (2.4) we may identify the arrow marked j^* with the linear dual of the arrow marked j_*. Simple evaluation according to (2.4) will be written

$$< T, l > , \quad T \in \Gamma_c(X,)^{\cdot \vee}), \quad l \in \Gamma(X, \Omega^{\cdot \vee \vee}) .$$

This notation is compatible with the symbol introduced in section 1 taking the biduality morphism (2.6) into account. We leave the remaining details with the reader. Q.E.D.

6. Poincaré duality

Let X be a n-dimensional *oriented* smooth manifold. A compactly supported $(n-p)$-form α on X defines a compact p-chain $P\alpha$ given by

$$(6.1) \qquad < P\alpha, \beta > \; = \int_X \alpha \wedge \beta , \quad \beta \in \Gamma(X, \Omega^p) .$$

(6.2) Theorem. *For a smooth oriented n-dimensional manifold* X, *the transformation* P *induces an isomorphism*

$$P : H_c^{n-p}(X, \mathbf{C}) \to H_p^c(X, \mathbf{C}) , \quad p \in \mathbf{N} ,$$

from de Rham cohomology with compact support to de Rham homology.

Proof. The following diagram is commutative

$$(6.3) \qquad \begin{array}{ccc} \Gamma_c(X, \Omega^{n-p}) & \xrightarrow{P} & D_p^c(X, \mathbf{C}) \\ \downarrow (-1)^n d & & \downarrow (-1)^{p-1} b \\ \Gamma_c(X, \Omega^{n-p+1}) & \xrightarrow{P} & D_{p-1}^c(X, \mathbf{C}) \end{array}$$

as it follows from the relation

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{n-p} \alpha \wedge d\beta , \quad \alpha \in \Gamma_c(X, \Omega^{n-p}), \beta \in \Gamma(X, \Omega^p) ,$$

using that $\int d(\alpha \wedge \beta) = 0$. Upon replacing X by an arbitrary open subset we obtain a morphism of complexes of sheaves

$$P: \Omega^{\bullet}[n] \to \Omega^{\bullet\vee} \tag{6.4}$$

with the signs of the differentials modified according to the commutative diagram (6.3). The morphism (6.4) is a quasi-isomorphism as it follows by checking the case $X = \mathbf{R}^n$ by means of the Poincaré lemma with and without compact support. As in the proof of (2.1) we conclude that P induces a quasi-isomorphism

$$P: \Gamma_c(X, \Omega^{\bullet}[n]) \to \Gamma_c(X, \Omega^{\bullet\vee}).$$

The second complex may be identified with $D^c_{\bullet}(X, \mathbf{C})$ as we have seen in (2.3) and the result follows by passing to homology. Q.E.D.

Let us extend Poincaré duality to the relative groups of an open subset U of X with complement Z in X. With the notation of (6.1), the operator P from (6.4) induces a commutative diagram

$$\begin{array}{ccccccc}
0 \to \Gamma_c(U, \Omega^{\bullet}[n]) & \to & \Gamma_c(X, \Omega^{\bullet}[n]) & \to & \Gamma_c(Z, \Omega^{\bullet}[n]) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow P & & \\
0 \to D^c_{\bullet}(U, \mathbf{C}) & \overset{j_*}{\to} & D^c_{\bullet}(X, \mathbf{C}) & \to & D^c_{\bullet}(X, U; \mathbf{C}) & \to & 0.
\end{array}$$

Again the differentials in the bottom row must be modified as in (6.3). The unmarked vertical arrows are the quasi-isomorphisms of Poincaré duality. The vertical arrow marked P is induced by the algebra of the diagram. Again, from algebra we deduce a *quasi-isomorphism*

$$P: \Gamma_c(Z, \Omega^{\bullet}[n]) \to D^c_{\bullet}(X, U; \mathbf{C}), \quad Z = X - U. \tag{6.5}$$

Passing to homology we obtain the Poincaré duality isomorphism

$$P: H^{n-p}_c(Z, \mathbf{C}) \xrightarrow{\sim} H^c_p(X, U; \mathbf{C}). \tag{6.6}$$

The p'th sheaf cohomology group with compact support $H^p_c(Z, \mathbf{C})$ has the following de Rham representation

$$\{\omega \in \Gamma_c(X, \Omega^p) \mid \operatorname{Supp}(d\omega) \subseteq U\} \Big/ \begin{array}{l} \{d\nu \mid \nu \in \Gamma_c(X, \Omega^{p-1})\} \\ + \{\omega \in \Gamma_c(X, \Omega^p) \mid \operatorname{Supp}(\omega) \subseteq U\} \end{array} \tag{6.7}$$

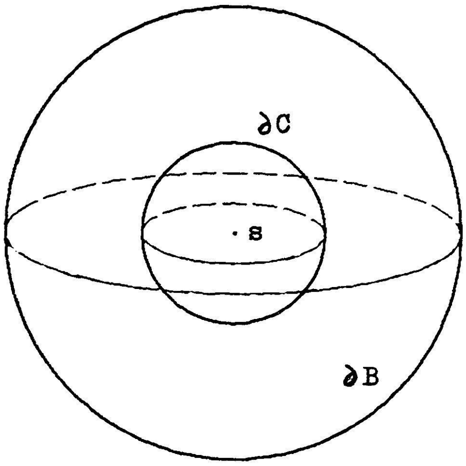
as it follows from the exact sequence making up the top row of the diagram above.

7. Winding numbers

Let X be an n-dimensional oriented smooth manifold and s a point of X. Consider a compact n-dimensional submanifold with boundary B with s as an interior point and put

$$\operatorname{Tr}(\omega\,;s) = \int_{\partial B} \omega\,, \quad \omega \in \Gamma(X-\{s\}, \Omega^{n-1}),\, d\omega = 0\,. \tag{7.1}$$

This symbol is independent of B as it follows by considering a small "ball" C around s contained in the interior of B



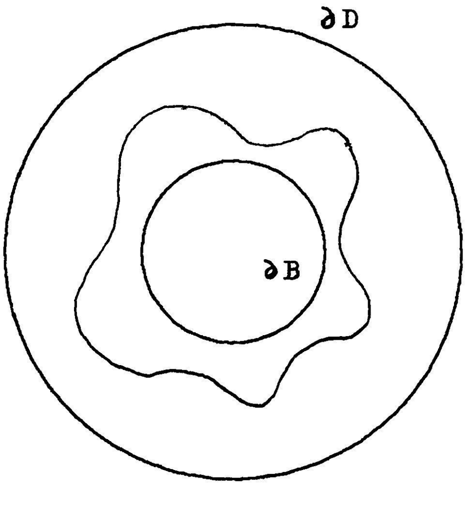
Stokes formula for $B - C^0$

$$\int_{\partial B} \omega - \int_{\partial C} \omega = \int_{B-C^0} d\omega\,.$$

Alternatively, choose a compactly supported smooth real function ρ_s on X which is constant 1 in a neighbourhood of s. Then

$$\operatorname{Tr}(\omega\,;s) = (-1)^n \int_X \omega \wedge d\rho_s\,, \quad \omega \in \Gamma(X-\{s\}, \Omega^{n-1}),\, d\omega = 0\,. \tag{7.2}$$

Proof. Choose "balls" B and D with center s such that ρ_s is constant 1 on B while $\operatorname{Supp}(\rho_s)$ is contained in the interior of D. From Stokes formula we get that



$$\int_{\partial D} \rho_s\omega - \int_{\partial B} \rho_s\omega = \int_{D-B} \rho_s \wedge \omega - \int_{D-B} \rho_s\omega$$

$$- \int_{\partial B} \omega = -(-1)^n \int_X \omega \wedge d\rho_s + \int_{D-B} \rho_s d\omega\,.$$

Notice that the last terms vanishes in case ω is exact. Q.E.D.

(7.3) *Example.* Let E denote an oriented n-dimensional Euclidian space. The distance r to the origin defines a 1-form dr^{2-n} on $E - \{0\}$. The dual form $*dr^{2-n}$ in the sense of Hodge is closed with

$$\mathrm{Tr}\,(*dr^{2-n}; 0) = (2-n)\sigma_{n-1}$$

where σ_{n-1} denotes the area of the unit sphere in E, compare [3] VII. 1.

Let us interprete (7.1) in terms of de Rham homology. Integration of n-forms on X over the manifold B determines a compact n-chain on X whose boundary, as written in (7.1), has support in $X - \{s\}$. The corresponding relative homology class

$$\theta_s \in H_n^c(X, X-\{s\}; \mathbf{C}), \quad s \in X, \tag{7.4}$$

is independent of B: with the notation above, the compact n-chain $\int_B - \int_C$ has support in $X - \{s\}$. The relative homology class we have just constructed is often called *the local orientation class.*

(7.5) PROPOSITION. *Let s be a point of an oriented n-dimensional smooth manifold X. The local orientation class θ_s generates $H_n^c(X, X-\{s\}; \mathbf{C})$.*

Proof. With the terminology from section 5 we may express formula (7.1) by means of the local orientation class

$$\mathrm{Tr}\,(\omega; s) = <\theta_s, \partial\omega> = <b\theta_s, \omega>, \quad \omega \in H^{n-1}(X-\{s\}, \mathbf{C}). \tag{7.6}$$

In case $n > 2$ we conclude from (7.3), that $\theta_s \neq 0$. The case $n = 2$ is left with the reader. Q.E.D.

Let us remark that formula (7.2) shows how to identify θ_s under relative Poincaré duality (6.6).

(7.7) PROPOSITION. *Let S be a finite subset of the oriented n-dimensional compact manifold X. For any closed form $\omega \in \Gamma(X-S, \Omega^n)$ we have that*

$$\sum_{s \in S} \mathrm{Tr}\,(\omega; s) = 0.$$

Proof. Let the *fundamental class* $\theta \in H_n(X, \mathbf{C})$ be given by

$$<\theta, \omega> = \int_X \omega, \quad \omega \in \Gamma(X, \Omega^n).$$

Let us consider a point $s \in S$ and use the notation from (7.1). The difference $\int_X - \int_B$ has support in $X - \{s\}$, which shows that the image of θ in $H_n(X, X-\{s\}; \mathbf{C})$ is θ_s. We have that

$$\sum_{s \in S} \mathrm{Tr}(\omega; s) = \sum_{s \in S} <\theta_s, \partial\omega> = <\theta_S, \partial\omega> = <b\theta_S, \omega>$$

where θ_S denotes the restriction of θ to $H_n(X, X-S; \mathbf{C})$. Conclusion by the fact that $b\theta_S = 0$. Q.E.D.

(7.8) *Definition.* Let γ be a compact n-chain on the oriented n-dimensional smooth manifold X. For a point $s \in X$ outside $\mathrm{Supp}(b\gamma)$ the class of γ in $H_n^c(X, X-\{s\}; \mathbf{C})$ can be written

$$[\gamma] = \mathrm{Ind}(\gamma; s)\theta_s, \quad \mathrm{Ind}(\gamma; s) \in \mathbf{C}.$$

The number $\mathrm{Ind}(\gamma; s)$ is called the *winding number* of γ with respect to s.

(7.9) *Example.* Let K denote an n-dimensional compact submanifold with boundary. Integration over K defines a compact n-chain κ with $\mathrm{Supp}(\partial\kappa) = \partial K$. The winding number for κ is 1 in the interior of K and 0 outside K.

(7.10) THEOREM. *Let γ be a compact n-chain on the oriented n-dimensional smooth manifold X. The winding number $s \mapsto \mathrm{Ind}(\gamma; s)$ is a locally constant function on the complement of $\mathrm{Supp}(b\gamma)$ in X. This function is zero outside some compact subset of X containing $\mathrm{Supp}(b\gamma)$.*

Proof. Let us consider an arbitrary open subset U of X containing $\mathrm{Supp}(b\gamma)$. We shall now use relative Poincaré duality to describe the class of γ in $H_n^c(X, U; \mathbf{C})$. According to (6.6) and (6.7) we can represent γ by a relative n-chain of the form

$$<\gamma, \nu> = \int_X \rho\nu, \quad \nu \in \Gamma(X, \Omega^n)$$

where ρ is a compactly supported smooth function on X, constant in a neighbourhood of any point s of $Z = X - U$. Let us notice that

$$<\partial\gamma, \omega> = (-1)^n \int \omega \wedge d\rho, \quad \omega \in \Gamma(U, \Omega^{n-1}), \ d\omega = 0.$$

In order to calculate $\mathrm{Ind}(\gamma; s)$ we replace U by a small pointed neighbourhood D^* of s. With the notation of (7.2) let us write $\rho = \rho(s)\rho_s$ and deduce that

$$<\partial\gamma, \omega> = \rho(s)\mathrm{Tr}(\omega; s), \quad \omega \in \Gamma(D^*, \Omega^{n-1}), \ d\omega = 0.$$

We can now conclude from (7.6) that

$$\mathrm{Ind}(\gamma; s) = \rho(s), \quad s \in X - U.$$

This reveals that $s \mapsto \mathrm{Ind}(\gamma; s)$ is a compactly supported, locally constant function on $X - U$.

For a given fixed point $s \notin \mathrm{Supp}(b\gamma)$ choose U to be an open neighbourhood of $\mathrm{Supp}(b\gamma)$ with $\bar{U}$ compact and $s \notin U$. We can apply the considerations above and conclude that the winding number is constant in a neighbourhood of s and zero outside some compact neighbourhood of $\mathrm{Supp}(b\gamma)$. Q.E.D.

(7.11) COROLLARY. *Let* γ *be a compact n-chain on the oriented smooth manifold* X *and* U *an open subset of* X *containing* $\mathrm{Supp}(b\gamma)$. *The relative de Rham homology class*

$$[\gamma] \in H_n^c(X, U; \mathbf{C})$$

is zero if and only if $\mathrm{Ind}(\gamma; s) = 0$ *for all* $s \in X - U$.

Proof. This is a corollary to the proof of (7.10) rather than the statement (7.10). Anyway, the basic point is Poincaré duality (6.6). Q.E.D.

8. CAUCHY'S RESIDUE THEOREM

We shall consider a smooth map $\gamma: S^{n-1} \to E$ from the oriented $n - 1$ sphere into an oriented n-dimensional real vector space E. For a point s outside $\gamma(S^{n-1})$ pick a closed $(n-1)$-form ω_s on $E - \{s\}$ with $\mathrm{Tr}(\omega_s; s) = 1$ and define the *winding number* of γ with respect to s to be

(8.1) $$\mathrm{Ind}(\gamma; s) = \int_{S^{n-1}} \gamma^*\omega_s .$$

(8.2) CAUCHY'S RESIDUE THEOREM. *Let* $\gamma: S^{n-1} \to X$ *denote a smooth map into an open subset* X *of* E *with* $\mathrm{Ind}(\gamma; z) = 0$ *for all* $z \in E - X$.

For a closed and discrete subset S *of* X *disjoint from* $\gamma(S^{n-1})$ *only finitely many of the numbers* $\operatorname{Ind}(\gamma; s)$, $s \in S$, *are distinct from zero and*

$$\int_{S^{n-1}} \gamma^*\omega = \sum_{s \in S} \operatorname{Ind}(\gamma; s) \operatorname{Tr}(\omega; s)$$

for any closed form ω *on* $X - S$.

Proof. The long exact de Rham homology sequence for the pair $X - S, E$ degenerates into an isomorphism

$$b\colon H_n^c(E, X-S; \mathbf{C}) \xrightarrow{\sim} H_{n-1}^c(X-S, \mathbf{C}) .$$

Let us view γ as a homology class on $X - S$ and introduce the class

$$b^{-1}\gamma \in H_n^c(E, X-S; \mathbf{C}) .$$

Let us notice that the winding number (8.1) and (7.8) agree. Thus we conclude from (7.11) that $b^{-1}\gamma$ maps to zero in $H_n^c(E, X; \mathbf{C})$ and consequently that γ is homologous to zero on X. The exact sequence

$$0 \to H_n^c(X, X-S; \mathbf{C}) \xrightarrow{b} H_{n-1}^c(X-S, \mathbf{C}) \to H_{n-1}^c(X, \mathbf{C})$$

allows us to interpret γ as a relative class

$$\gamma \in H_n^c(X, X-S; \mathbf{C}) .$$

The class γ can be specified by the formula

$$\langle b\gamma, \omega \rangle = \int_{S^{n-1}} \gamma^*\omega , \quad \omega \in \Gamma(X-S, \Omega^{n-1}) , \ d\omega = 0 .$$

From the decomposition (4.9) and excision (4.6) we deduce a canonical isomorphism

$$H_n(X, X-S; \mathbf{C}) \xrightarrow{\sim} \bigoplus_{s \in S} H_n(X, X-\{s\}; \mathbf{C})$$

which allow us to decompose the class γ into a finite sum, compare (7.6),

$$\gamma = \sum_{s \in S} \operatorname{Ind}(\gamma; s)\theta_s .$$

Using the general Stokes formula (5.3) we get that

$$\langle b\gamma, \omega \rangle = \langle \gamma, \partial\omega \rangle = \sum \operatorname{Ind}(\gamma; s) \langle \theta_s, \partial\omega \rangle$$

and the result follows from formula (7.6). Q.E.D.

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