

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 35 (1989)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE FIXED POINT SET OF A FINITE GROUP ACTION ON A
HOMOLOGY FOUR SPHERE
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Kapitel: 5. LOCALLY LINEAR REPRESENTATION
DOI: <https://doi.org/10.5169/seals-57368>

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$\text{Fix}(A_4)$ is a sphere. It cannot be S^2 since the representation of A_4 in $SO(3)$ is irreducible, so it is S^1 . The only closed 1-dimensional submanifold of S^1 is S^1 itself, so $\text{Fix}(G) = S^1$.

b. As in subcase a., a linear change in coordinates allows us to assume that h is actually \tilde{i} , and as before if $G_2 \in G$ the proposition is proved applying 4.1.

If it is not the case, let α correspond to the cycle $(12345) \in A_5$, β to (123) and γ to (345) . We observe that β and γ generate A_5 and so:

1. $\text{Fix}(A_5) = \text{Fix}(\beta) \cap \text{Fix}(\gamma)$,
2. $\text{Fix}(A_5) \subset \text{Fix}(\alpha)$.

We claim that $\text{Fix}(\alpha)$ is S^0 . According to Smith's theorem it is enough to prove that the representation of α around x_0 has an isolated fixed point, i.e. is the sum of two irreducible complex ones.

If not by Lemma 3.3 $(\bar{i}(\alpha); i(\alpha))$ would be conjugate in $SO(3) \times SO(3)$ to an element on the diagonal. From the explicit description of i and \bar{i} (see the end of section 7.1 of [22]), it follows that they send all the five cycles to non conjugate elements in $SO(3)$, so this is impossible, and $\text{Fix}(\alpha) = S^0$.

As for β and γ , their images under (\bar{i}, i) are conjugate to elements on the diagonal, by 3.3 and 3.4 their fixed point sets have two-dimensional components, and so by Smith's theorem they are copies of S^2 .

So $\text{Fix}(G)$ is the intersection of a couple of S^2 s and is contained in $\text{Fix}(\alpha)$ which is S^0 . If this set is empty or equal to S^0 , the proposition follows. If it were a single point, it would be a transverse intersection, by local linearity, but it is not possible since a homology S^4 does not contain any two cycles with intersection number odd. This ends the proof.

5. LOCALLY LINEAR REPRESENTATION

Let's now consider the case of G acting on a homology S^4 with two fixed points, P_0 and P_1 .

THEOREM 5.1. *The unoriented representations of G around P_0 and P_1 are linearly equivalent.* ¹⁾

Proof. It will suffice to show that the characters associated to the representations around the P_i s agree on every cyclic subgroup C_k of G .

¹⁾ See the note in the introduction.

Observe that by Lemma 3.4 and Smith's theorem the fixed point set of an element of G different from the identity is either S^0 or S^2 .

Let g generate C_k , we distinguish three cases:

1. $\text{Fix}(g^r) = \{P_1; P_2\}$ for every $r \equiv 0 \pmod{k}$,
2. $\text{Fix}(g) = S^2$,
3. $\text{Fix}(g) = \{P_1; P_2\}$ but $\text{Fix}(g^n) = S^2$ for some $g^n \neq id$.

Case 1. The hypothesis means that the action is semifree and the claim follows from the work of Atiyah and Bott, see [1] and [14].

Case 2. The action of C_k on the normal bundle of the fixed S^2 defines an element N of $K_{C_k}(S^2)$. Since C_k acts trivially on S^2 the two inclusions $P_i \rightarrow S^2$ are obviously C_k homotopic so that the diagram:

$$\begin{array}{ccccc} & & K_{C_k}(P_2) & & \\ & \nearrow & & \searrow & \\ [N] \in K_{C_k}(S^2) & & & & R(C_k) \\ & \searrow & & \nearrow & \\ & & K_{C_k}(P_1) & & \end{array}$$

commutes. This means that the representation of C_k in the normal component to S^2 are conjugate, the tangential representations are of course both the identity, so the statement is proved.

Case 3. We can assume, by [8], that the action on $S^2 = \text{Fix}(g^n)$ is linear. S^2 has zero intersection number in Σ so its normal bundle N can be identified to $S^2 \times R^2$, and we fix a trivialization. Denote a point of $S^2 - \{P_1; P_2\}$ by (x, t) with $x \in S^1$ and $t \in (0, 1)$. Let C_0 be the space $\{\phi: S^1 \rightarrow SO(2) \mid \deg \phi = 0\}$, it is an abelian group by pointwise multiplication and a C_k module with structure given by:

$$(h\phi)(x) = \phi(hx), h \in C_k \quad \text{and} \quad x \in S^1 \subset S^2$$

acted on by the obvious induced action.

By [5], chapter VI, prop. 11.1, the action is given by a θ_t such that

1. $\theta_t \in Z^1(C_k; C_0)$ and depends continuously on $t \in [0, 1]$.
2. $\theta_i(h)(x)$ is constant on $x \in S^1$ and equal to the representation of h at P_i for $i = 0; 1$.

A change in the trivialization adds to each θ_t a coboundary so there is a well defined continuous family $\theta_t: [0, 1] \rightarrow H^1(C_k; C_0)$.

A straightforward calculation shows that $H^1(C_k; C_0) = H^2(C_k; Z) = C_k$. Since θ_t is continuous it has to be constant, so $\theta_0 = \theta_1$ and by 2. the

two normal representations are equal. In the topological case, by the results of Cappel and Shaneson topological equivalence of matrices in dimension 4 implies linear equivalence, so the statement of Theorem 5.1 makes sense also for a group of homeomorphism.

The proof given can be adapted to this more general case provided that the followings are true:

1. the topological Atiyah-Singer signature formula holds,
2. a locally flat S^2 in Σ has a normal bundle,
3. the argument in case 3 works with $\text{Homeo}(S^1)$ instead of $SO(2)$.

Assertion 1 is proved, in the case of the semi-free action, in [21], page 188; assertion 2 follows from the work of Freedman, see [10]; assertion 3 is proved using the retraction $\text{Homeo}(S^1)$ into $SO(2)$ given by the Poincaré number, see [7].

APPENDIX

LEMMA. *The extensions:*

$$\begin{array}{ccccccc}
 0 & \rightarrow & C_2 & \rightarrow & \tilde{A}_5 & \rightarrow & A_5 \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 0 & \rightarrow & C_2 & \rightarrow & A_5 \times A_5 & \rightarrow & A_5 \times A_5 \rightarrow 0 \\
 & & & & \downarrow & & \downarrow (h, h') \\
 0 & \rightarrow & C_2 & \rightarrow & SO(4) & \rightarrow & SO(3) \times SO(3) \rightarrow 0
 \end{array}$$

are not split, h and h' can be any nontrivial representations of A_5 and f is either $(Id \times \{I\})$ or $(\{I\} \times Id)$.

Proof. Standard theory of group extensions and cohomology (see [4]) allows us to reduce to the:

PROPOSITION. Any non trivial homomorphism $A_5 \xrightarrow{i} SO(3)$ induces an isomorphism $Z/2 = H^2(BSO(3); Z/2) \xrightarrow{i} H^2(BA_5; Z/2) = Z/2$.

Proof of the Proposition. If the corresponding extension is split, then $Z/2 \times A_5 \subset S^3$, but $A_5 = 60$ so there exists a $Z/2 \subset A_5$ so $Z/2 \times Z/2$ would act freely on S^3 , which cannot happen.