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HOMOLOGY FOUR SPHERE  
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products, subgroups and central extensions, so  $G$  falls in the hypothesis of Corollary 3.1. in all cases, except the one in which at least one  $G_i$  is the icosahedral group. This is isomorphic to  $A_5$ , the alternating group on five letters and this identification will be fixed from now on.

#### 4. NON SOLVABLE GROUPS

We will prove Theorem 2.1 case by case. We start with the Lemma:

LEMMA 4.1. *If  $G$  contains  $C_2$ , then  $\text{Fix}(G)$  is  $S^0$ .*

*Proof.*  $\text{Fix}(G) = \text{Fix}(G/C_2\text{Fix}(C_2))$ .  $\text{Fix}(C_2)$  is a homology sphere by Smith's theorem and is zero dimensional since around the chosen fixed point the non trivial element of  $C_2$  acts like the matrix  $-I$ , which has an isolated fixed point. The action of  $G/C_2$  on  $S^0$  has to be trivial since the fixed point set is required not to be empty.

By renumbering the factors and changing basis if necessary, we may assume  $G_2$  equal to  $A_5$ , with  $G_2 \xrightarrow{i} SO(3)$  the standard representation of  $A_5$ . Then  $G_0$  is a subgroup of  $G_1 \times A_5$  mapping onto both factors and to study it in more detail we look at the kernel of the second projection:  $G_0 \xrightarrow{\pi_2} A_5$ . This subgroup consists of elements of the form  $(k, I)$  with  $k \in G_1$ ; we denote it by  $K_1$ .

For convenience we distinguish three cases:

Case 1.  $K_1$  is a non-trivial subgroup of  $SO(3)$ , not isomorphic to  $A_5$ ,

Case 2.  $K_1$  is isomorphic to  $A_5$ ,

Case 3.  $K_1$  is trivial.

*Proof in case 1.* The surjection  $G \rightarrow A_5$  has non trivial kernel  $K = j^{-1}(\pi^{-1}(K_1)) \subset G$ , this group is solvable since  $K_1$  is,  $\pi$  is a central extension and  $j$  is an injection. By Corollary 3.1.,  $\text{Fix}(K)$  is a sphere of dimension 2 and  $\text{Fix}(G)$  is the fixed point set of an  $A_5$  acting on it, so it is easy to see that the only actions admitting some fixed points are the trivial ones.

*Proof in case 2.* Since  $A_5$  is not properly contained in any finite subgroup of  $SO(3)$ ,  $K_1$  has to be equal to the whole  $G_1$ .

So  $G_0 \subset A_5 \times A_5 \subset SO(3) \times SO(3)$  and contains  $K_1 = A_5 \times \{I\}$ , it follows that  $G_0$  is the whole  $A_5 \times A_5$ . Observe that the two inclusions of  $A_5$  in  $SO(3)$  do not necessarily agree.

We claim that  $G$  in the diagram 3.5 must contain  $C_2$ , for if not  $j \circ \pi$  would be an isomorphism  $G \rightarrow A_5 \times A_5$  and its inverse would split the extension

$$0 \rightarrow C_2 \rightarrow \tilde{G} \rightarrow A_5 \rightarrow A_5 \rightarrow 0$$

This is not possible (see the appendix). Now apply Lemma 4.1. to end the proof.

*Proof in case 3.* If  $K_1$  is trivial the projection  $G_0 \xrightarrow{\pi_2} A_5$  is an isomorphism and the composition  $\phi = \pi_1 \circ \pi_2^{-1}: A_5 \rightarrow G_1$  is a map onto, with graph  $G_0$ . The homomorphic images of  $A_5$  are only the trivial group and  $A_5$  itself, since  $A_5$  is simple.

If  $G_1 = \phi(A_5)$  is trivial,  $G_0$  is equal to  $\{I\} \times A_5$ . As in case 2 the extension

$$0 \rightarrow C_2 \rightarrow G \rightarrow \{I\} \times A_5$$

is not split, so  $G$  contains  $C_2$  and  $\text{Fix}(G) = S$  by 4.1. If  $G_1 = \phi(A_5)$  is isomorphic to  $A_5$ ,  $G_0 \subset G_1 \times G_2$  is a copy of  $A_5$  too, mapped into  $SO(3) \times SO(3)$  according to  $d(x) = (h(x); i(x))$ , where  $h(x)$  is some irreducible representation and  $i(x)$  is the standard one specified before. The arguments in [22] can be used to prove that there are exactly two equivalence classes of representations of  $A_5$  into  $SO(3)$ .

So there are two subcases:

- a.  $h$  is  $x \rightarrow u^{-1}i(x)u$ , with  $u \in SO(3)$ ,
- b.  $h$  is conjugate to the composition  $\bar{i}: A_5 \xrightarrow{\sigma} A_5 \xrightarrow{i} SO(3)$  and  $\sigma$  is conjugation by the cycle  $(\bar{i}_2)S_5$  on  $A_5$ .

a. If the coordinate system around the fixed point chosen at the beginning is linearly changed according to some  $\tilde{u} \in SO(4)$ , the representation  $\rho: G \rightarrow SO(4)$  becomes  $\tilde{u}\rho(x)\tilde{u}^{-1}$ .

If  $\pi(\tilde{u}) = (u; 1)$ ;  $i$  is left unchanged and  $h$  is replaced by  $i$ . So  $G_0$  is contained in the diagonal and  $G \in \tilde{G} \in \text{Im}(O(3))$ .

Recall that when  $G$  contains  $C_2$ ,  $\text{Fix}(G) = S^0$  by Lemma 4.1.

LEMMA 4.2. If  $G \neq C_2$ ,  $\text{Fix}(G) = S^1$ .

*Proof.*  $G$  is isomorphic to  $A_5$  and has to be contained in  $\text{Im}(SO(3))$  so its representation has a one dimensional fixed space, which implies  $\text{Fix}(G)$  1-dimensional at  $x_0$ . Now  $A_5$  contains  $A_4$  (named tetrahedral group when sitting in  $SO(3)$ ), so  $\text{Fix}(A_5) \in \text{Fix}(A_4)$ ,  $A_4$  is solvable and hence

$\text{Fix}(A_4)$  is a sphere. It cannot be  $S^2$  since the representation of  $A_4$  in  $SO(3)$  is irreducible, so it is  $S^1$ . The only closed 1-dimensional submanifold of  $S^1$  is  $S^1$  itself, so  $\text{Fix}(G) = S^1$ .

b. As in subcase a., a linear change in coordinates allows us to assume that  $h$  is actually  $\tilde{i}$ , and as before if  $G_2 \in G$  the proposition is proved applying 4.1.

If it is not the case, let  $\alpha$  correspond to the cycle  $(12345) \in A_5$ ,  $\beta$  to  $(123)$  and  $\gamma$  to  $(345)$ . We observe that  $\beta$  and  $\gamma$  generate  $A_5$  and so:

1.  $\text{Fix}(A_5) = \text{Fix}(\beta) \cap \text{Fix}(\gamma)$ ,
2.  $\text{Fix}(A_5) \subset \text{Fix}(\alpha)$ .

We claim that  $\text{Fix}(\alpha)$  is  $S^0$ . According to Smith's theorem it is enough to prove that the representation of  $\alpha$  around  $x_0$  has an isolated fixed point, i.e. is the sum of two irreducible complex ones.

If not by Lemma 3.3  $(\bar{i}(\alpha); i(\alpha))$  would be conjugate in  $SO(3) \times SO(3)$  to an element on the diagonal. From the explicit description of  $i$  and  $\bar{i}$  (see the end of section 7.1 of [22]), it follows that they send all the five cycles to non conjugate elements in  $SO(3)$ , so this is impossible, and  $\text{Fix}(\alpha) = S^0$ .

As for  $\beta$  and  $\gamma$ , their images under  $(\bar{i}, i)$  are conjugate to elements on the diagonal, by 3.3 and 3.4 their fixed point sets have two-dimensional components, and so by Smith's theorem they are copies of  $S^2$ .

So  $\text{Fix}(G)$  is the intersection of a couple of  $S^2$ s and is contained in  $\text{Fix}(\alpha)$  which is  $S^0$ . If this set is empty or equal to  $S^0$ , the proposition follows. If it were a single point, it would be a transverse intersection, by local linearity, but it is not possible since a homology  $S^4$  does not contain any two cycles with intersection number odd. This ends the proof.

## 5. LOCALLY LINEAR REPRESENTATION

Let's now consider the case of  $G$  acting on a homology  $S^4$  with two fixed points,  $P_0$  and  $P_1$ .

**THEOREM 5.1.** *The unoriented representations of  $G$  around  $P_0$  and  $P_1$  are linearly equivalent.* <sup>1)</sup>

*Proof.* It will suffice to show that the characters associated to the representations around the  $P_i$ s agree on every cyclic subgroup  $C_k$  of  $G$ .

<sup>1)</sup> See the note in the introduction.