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Artikel: THE FIXED POINT SET OF A FINITE GROUP ACTION ON A HOMOLOGY FOUR SPHERE

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products, subgroups and central extensions, so G falls in the hypothesis of Corollary 3.1. in all cases, except the one in which at least one G_i is the icosahedral group. This is isomorphic to A_5, the alternating group on five letters and this identification will be fixed from now on.

4. Non solvable groups

We will prove Theorem 2.1 case by case. We start with the Lemma:

LEMMA 4.1. *If G contains C_2, then* $\mathrm{Fix}(G)$ *is* S^0.

Proof. $\mathrm{Fix}(G) = \mathrm{Fix}\big(G/C_2\mathrm{Fix}(C_2)\big)$. $\mathrm{Fix}(C_2)$ is a homology sphere by Smith's theorem and is zero dimensional since around the chosen fixed point the non trivial element of C_2 acts like the matrix $-I$, which has an isolated fixed point. The action of G/C_2 on S^0 has to be trivial since the fixed point set is required not to be empty.

By renumbering the factors and changing basis if necessary, we may assume G_2 equal to A_5, with $G_2 \xrightarrow{i} SO(3)$ the standard representation of A_5. Then G_0 is a subgroup of $G_1 \times A_5$ mapping onto both factors and to study it in more detail we look at the kernel of the second projection: $G_0 \xrightarrow{\pi_2} A_5$. This subgroup consists of elements of the form (k, I) with $k \in G_1$; we denote it by K_1.

For convenience we distinguish three cases:

Case 1. K_1 is a non-trivial subgroup of $SO(3)$, not isomorphic to A_5,

Case 2. K_1 is isomorphic to A_5,

Case 3. K_1 is trivial.

Proof in case 1. The surjection $G \to A_5$ has non trivial kernel $K = j^{-1}(\pi^{-1}(K_1)) \subset G$, this group is solvable since K_1 is, π is a central extension and j is an injection. By Corollary 3.1., $\mathrm{Fix}(K)$ is a sphere of dimension 2 and $\mathrm{Fix}(G)$ is the fixed point set of an A_5 acting on it, so it is easy to see that the only actions admitting some fixed points are the trivial ones.

Proof in case 2. Since A_5 is not properly contained in any finite subgroup of $SO(3)$, K_1 has to be equal to the whole G_1.

So $G_0 \subset A_5 \times A_5 \subset SO(3) \times SO(3)$ and contains $K_1 = A_5 \times \{I\}$, it follows that G_0 is the whole $A_5 \times A_5$. Observe that the two inclusions of A_5 in $SO(3)$ do not necessarily agree.

We claim that G in the diagram 3.5 must contain C_2, for if not $j \circ \pi$ would be an isomorphism $G \to A_5 \times A_5$ and its inverse would split the extension

$$0 \to C_2 \to \tilde{G} \to A_5 \to A_5 \to 0$$

This is not possible (see the appendix). Now apply Lemma 4.1. to end the proof.

Proof in case 3. If K_1 is trivial the projection $G_0 \xrightarrow{\pi_2} A_5$ is an isomorphism and the composition $\phi = \pi_1 \circ \pi_2^{-1}: A_5 \to G_1$ is a map onto, with graph G_0. The homomorphic images of A_5 are only the trivial group and A_5 itself, since A_5 is simple.

If $G_1 = \phi(A_5)$ is trivial, G_0 is equal to $\{I\} \times A_5$. As in case 2 the extension

$$0 \to C_2 \to G \to \{I\} \times A_5$$

is not split, so G contains C_2 and $\operatorname{Fix}(G) = S$ by 4.1. If $G_1 = \phi(A_5)$ is isomorphic to A_5, $G_0 \subset G_1 \times G_2$ is a copy of A_5 too, mapped into $SO(3) \times SO(3)$ according to $d(x) = (h(x); i(x))$, where $h(x)$ is some irreducible representation and $i(x)$ is the standard one specified before. The arguments in [22] can be used to prove that there are exactly two equivalence classes of representations of A_5 into $SO(3)$.

So there are two subcases:

a. h is $x \to u^{-1}i(x)u$, with $u \in SO(3)$,

b. h is conjugate to the composition $\bar{i}: A_5 \xrightarrow{\sigma} A_5 \xrightarrow{i} SO(3)$ and σ is conjugation by the cycle $(\bar{i}_2)S_5$ on A_5.

a. If the coordinate system around the fixed point chosen at the beginning is linearly changed according to some $\tilde{u} \in SO(4)$, the representation $\rho: G \to SO(4)$ becomes $\tilde{u}\rho(x)\tilde{u}^{-1}$.

If $\pi(\tilde{u}) = (u; 1)$; i is left unchanged and h is replaced by i. So G_0 is contained in the diagonal and $G \in \tilde{G} \in Im(O(3))$.

Recall that when G contains C_2, $\operatorname{Fix}(G) = S^0$ by Lemma 4.1.

LEMMA 4.2. *If* $G \neq C_2$, $\operatorname{Fix}(G) = S^1$.

Proof. G is isomorphic to A_5 and has to be contained in $Im(SO(3))$ so its representation has a one dimensional fixed space, which implies $\operatorname{Fix}(G)$ 1-dimensional at x_0. Now A_5 contains A_4 (named tetrahedral group when sitting in $SO(3)$), so $\operatorname{Fix}(A_5) \in \operatorname{Fix}(A_4)$, A_4 is solvable and hence

$\mathrm{Fix}(A_4)$ is a sphere. It cannot be S^2 since the representation of A_4 in $SO(3)$ is irreducible, so it is S^1. The only closed 1-dimensional submanifold of S^1 is S^1 itself, so $\mathrm{Fix}(G) = S^1$.

b. As in subcase a., a linear change in coordinates allows us to assume that h is actually $\tilde{i}$, and as before if $G_2 \in G$ the proposition is proved applying 4.1.

If it is not the case, let α correspond to the cycle $(12345) \in A_5$, β to (123) and γ to (345). We observe that β and γ generate A_5 and so:

1. $\mathrm{Fix}(A_5) = \mathrm{Fix}(\beta) \cap \mathrm{Fix}(\gamma)$,
2. $\mathrm{Fix}(A_5) \subset \mathrm{Fix}(\alpha)$.

We claim that $\mathrm{Fix}(\alpha)$ is S^0. According to Smith's theorem it is enough to prove that the representation of α around x_0 has an isolated fixed point, i.e. is the sum of two irreducible complex ones.

If not by Lemma 3.3 $(\bar{i}(\alpha); i(\alpha))$ would be conjugate in $SO(3) \times SO(3)$ to an element on the diagonal. From the explicit description of i *and* $\bar{i}$ (see the end of section 7.1 of [22]), it follows that they send all the five cycles to non conjugate elements in $SO(3)$, so this is impossible, and $\mathrm{Fix}(\alpha) = S^0$.

As for β and γ, their images under $(\bar{i}, i)$ are conjugate to elements on the diagonal, by 3.3 and 3.4 their fixed point sets have two-dimensional components, and so by Smith's theorem they are copies of S^2.

So $\mathrm{Fix}(G)$ is the intersection of a couple of S^2s and is contained in $\mathrm{Fix}(\alpha)$ which is S^0. If this set is empty or equal to S^0, the proposition follows. If it were a single point, it would be a transverse intersection, by local linearity, but it is not possible since a homology S^4 does not contain any two cycles with intersection number odd. This ends the proof.

5. Locally linear representation

Let's now consider the case of G acting on a homology S^4 with two fixed points, P_0 and P_1.

THEOREM 5.1. *The unoriented representations of G around P_0 and P_1 are linearly equivalent.* [1])

Proof. It will suffice to show that the characters associated to the representations around the P_is agree on every cyclic subgroup C_k of G.

[1]) See the note in the introduction.