Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 35 (1989)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE FIXED POINT SET OF A FINITE GROUP ACTION ON A HOMOLOGY FOUR SPHERE

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DOI: https://doi.org/10.5169/seals-57368

The aim of this paper is to prove that any locally linear orientation preserving action of a finite group on an homology four sphere has fixed point set homeomorphic to a sphere. In particular there are no one fixed point actions. Besides, if the fixed point set is S^0, it is proved that the local representations are conjugate.[1]) For a large class of actions, the proof is an elementary application of Smith's theory, using the fact that in dimensions $\leqslant 2$ homology spheres are topological spheres. In one remaining case, an action of the icosahedral group, a slightly more complicated argument is needed. This type of argument cannot be extended to dimension 3, as the example in [11] proves.

The motivation for this work came from the paper of Peter Braam and Gordana Matic [3] on group actions and instantons spaces. They prove that a smooth orientation preserving action of a group on a homology sphere whose fundamental group has no nontrivial representations in $SU(2)$ admits an even number of isolated fixed points and that they come in pair such that the representations around them are conjugate. Also, Furuta proved that there are no actions with one fixed point.

The author wishes to thank Professor William Browder for his patience in listening to him and for his advice, and also Gordana Matic for having explained her work to him.

2. Statement of the result

In the following "R-homology S^n" will mean a compact topological manifold whose homology with coefficients in the ring R is the same as that of S^n. (Of course in dimensions 0, 1, 2 such a manifold is homeomorphic to a sphere.) To unify some notation, the empty set will be considered a sphere of dimension -1, all actions will be assumed effective.

THEOREM 2.1. *Let G be a finite group acting locally linearly and preserving the orientation on a Z-homology 4-sphere Σ. Then the fixed point set of G is homeomorphic to a sphere; in particular it never consists of one point.*

Local linearity is assumed to avoid pathologies, every smooth action is locally linear (see e.g. [5]).

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[1]) The author has been informed that this has been proved independently by S. Cappell.

Observation 2.2. There is a non-locally linear action of a finite group on a homology four sphere with exactly one fixed point.

Proof. Take the one fixed point action of A_5 on the Poincaré's sphere constructed in [11], remove the fixed point and multiply the remaining homology disk by the unit interval to obtain a four homology disk on which the product action has no fixed points. One can extend this action to the one point compactification to obtain a homology S^4 on which A_5 acts fixing only the point at infinity.

The main tool in the proof of Theorem 2.1. will be the classical result due to Smith (see [19]);

THEOREM 2.3. *Let* Z/p, *p a prime, act on a* Z/p *homology* S^n, *then the fixed point set is a* Z/p *homology* S^k; *if* p *is odd,* $n - k$ *is even.*

3. SOLVABLE GROUPS

In the four dimensional case it is easy to deduce from Theorem 2.3. the Corollary:

COROLLARY 3.1. *Let* G *be a solvable group acting locally linearly and orientation preserving on* Σ, *then the fixed point set is a sphere.*

Proof of the Corollary. Let $\{ I \} = H_0 \subset H_1 \subset H_2 \subset G$ be a composition series such that every H_{i+1} is normal in H_i and the quotients are cyclic of prime order p_i. By Smith theorem $X = \mathrm{Fix}(H_i)$ is a Z/p homology sphere, the action is not trivial so X cannot be the whole Σ; nor can it be 3-dimensional, for otherwise some element of H_1 would interchange the two components of $\Sigma - X$ and so reverse the orientation. Hence X has to be of dimension less than or equal to 2 and so a topological sphere.

For $i > 1$, $\mathrm{Fix}(H_{i-1})$ is invariant under H_i and the latter's action factorizes through H_i/H_{i-1}, so $\mathrm{Fix}(H_i) = \mathrm{Fix}\big(H_{i-1}/H_i \mid \mathrm{Fix}(H_{i-1})\big)$; applying repeatedly the argument above and using the fact that now all the spaces involved are spheres, the statement follows.

If $x_0 \in \Sigma^G$, the fixed set of G on Σ, the assumption of local linearity gives a representation $G \xrightarrow{\rho} SO(4)$, faithful since G acts effectively, this allows us to think of G as a finite subgroup of $SO(4)$ and to study it we look at the central extension: