

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 35 (1989)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: HURWITZ-RADON MATRICES AND PERIODICITY MODULO 8
Autor: Eckmann, Beno
Kapitel: 3. The homotopy groups of U and O
DOI: <https://doi.org/10.5169/seals-57365>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 10.12.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

THEOREM 2.3. E_*^U is the polynomial ring $\mathbb{Z}[\rho_1]$.

2.4. THE RING E_*^O .

We denote by σ_s the generator of E_s^O ($= 0$ if $s \equiv 2, 4, 5, 6$ modulo 8; determined up to sign if $s \equiv 3, 7$ modulo 8 where $E_s^O = \mathbb{Z}$).

The generator $\rho_7 (= \rho_1^4) \in E_7^U$ can be given by a real ε -representation of degree 8 which we can use as generator $\sigma_7 \in E_7^O$. The ring homomorphism $\Phi: E_*^O \rightarrow E_*^U$ induced by the embedding $O \rightarrow U$, $\Phi(\sigma_7) = \rho_7$, is thus an isomorphism $E_7^O \cong E_7^U$. In E_*^O the degree of $\sigma_7 \sigma_s \in E_{s+8}^O$ is $16d_s^O = d_{s+8}^O$. Hence $\sigma_7 \sigma_s$ is irreducible, i.e., $= \pm \sigma_{s+8}$ for all s . In particular we can choose $\sigma_{15} = \sigma_7^2$, $\sigma_{23} = \sigma_7^3$, ..., $\sigma_{8r-1} = \sigma_7^r$.

PROPOSITION 2.4. The isomorphism $E_s^O \cong E_{s+8}^O$ can be given by the product with $\sigma_7 \in E_7^O$.

PROPOSITION 2.5. $\sigma_7 \in E_7^O$ generates a subring of E_*^O which is the polynomial ring $\mathbb{Z}[\sigma_7]$.

We further note that $\sigma_3 \in E_3^O$ is mapped by Φ to $2\rho_3 \in E_3^U$. From $\Phi(\sigma_3^2) = 4\rho_3^2 = 4\rho_7 = \Phi(4\sigma_7)$ we infer that $\sigma_3^2 = 4\sigma_7$. As for $\sigma_0 \in E_0^O$, it is of degree 1 and order 2, and $\sigma_0^2 \in E_1^O$ is of degree 2 and order 2, i.e., $\sigma_0^2 = \sigma_1$. Of course $\sigma_0^3 = 0$.

In summary:

THEOREM 2.6. E_*^O is the commutative ring, graded by $s+1$ for E_s^O , generated by $\sigma_0, \sigma_3, \sigma_7$ with the only relations $2\sigma_0 = 0$, $\sigma_0^3 = 0$, $\sigma_3^2 = 4\sigma_7$.

3. THE HOMOTOPY GROUPS OF U AND O

3.1. We will deal explicitly with the unitary case. The orthogonal case can be treated in almost exactly the same way; any additional arguments will be mentioned wherever necessary.

In the Introduction 0.1 we associated with a set of s unitary $n \times n$ HR-matrices, i.e., with an ε -representation of G_s , a map $f: S^s \rightarrow U$ of the s -sphere $S^s \subset \mathbb{R}^{s+1}$ into the infinite unitary group U via $U(n)$. Since conjugation is homotopic to the identity, equivalent representations yield homotopic maps f (in the orthogonal case, we have to observe that conjugation can be made with a matrix from the identity component). The map $\phi: D_s^U \rightarrow \pi_s(U)$ thus obtained is a homomorphism; indeed, homotopy group addition of f and f' in $\pi_s(U(n))$ can be replaced by multiplication in

$U(n)$; this is homotopic in $U(2n)$ to the map $\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}$, and on the other hand addition in D_s^U is defined through the direct sum of representations.

If the ε -representation is restricted from D_{s+1}^U , i.e., if the set of HR-matrices belongs to a set of $s+1$ HR-matrices, f extends to a map $S^{s+1} \rightarrow U$ and is thus nullhomotopic. The homomorphism ϕ therefore induces a homomorphism $E_s^U \rightarrow \pi_s(U)$, again written ϕ . The analogue $E_s^O \rightarrow \pi_s(O)$ will be denoted by ψ . The groups E_s^U and E_s^O are 0 or cyclic generated by irreducible ε -representations, i.e., by HR-matrices of minimal size. Our claim, Theorem A, can therefore be reformulated as follows.

THEOREM B. *The homomorphisms $\phi: E_s^U \rightarrow \pi_s(U)$ and $\psi: E_s^O \rightarrow \pi_s(O)$ are isomorphisms, $s = 0, 1, 2, \dots$.*

3.2. For small values of s the claim is easily checked.

Case U

$s = 1$: E_1^U can be generated by one HR-matrix $A_1 = (i)$. Thus

$$f(x_0, x_1) = (x_0 + ix_1) \in U(1)$$

if $x_0^2 + x_1^2 = 1$. This is a generator of $\pi_1(U(1)) \cong \pi_1(U) = \mathbf{Z}$.

$s = 3$: E_3^U is generated by 3 HR-matrices

$$A_1 = \begin{pmatrix} i & \\ & -i \end{pmatrix}, A_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, A_3 = \begin{pmatrix} & i \\ i & \end{pmatrix}.$$

Thus

$$f(x_0, x_1, x_2, x_3) = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \in SU(2)$$

if $\sum_{j=0}^3 x_j^2 = 1$. This is a generator of $\pi_3(SU(2)) [= \pi_3(S^3)] \cong \pi_3(U) = \mathbf{Z}$.

Case O

$s = 0$: Empty set of HR-matrices, $f(x_0) = (x_0) \in O(1)$ if $x_0^2 = 1$, $x_0 = \pm 1$. This is a generator of $\pi_0(O(1)) \cong \pi_0(O) = \mathbf{Z}/2$.

$s = 1$: E_1^O is generated by one HR-matrix $A_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. Thus

$$f(x_0, x_1) = \begin{pmatrix} x_0 & x_1 \\ -x_1 & x_0 \end{pmatrix} \in SO(2)$$

if $x_0^2 + x_1^2 = 1$. This is a generator of $\pi_1(SO(2)) = \mathbf{Z}$; as a map $S^1 \rightarrow SO(3)$ it is a generator of $\pi_1(SO(3)) \cong \pi_1(O) = \mathbf{Z}/2$.

$s = 3$: E_3^O is generated by three 4×4 HR-matrices which yield

$$f(x_0, x_1, x_2, x_3) = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_1 & x_0 & x_3 & -x_2 \\ -x_2 & -x_3 & x_0 & x_1 \\ -x_3 & x_2 & -x_1 & x_0 \end{pmatrix} \in SO(4)$$

if $\sum_0^3 x_j^2 = 1$. This is a map $S^3 \rightarrow SO(4)$ which is well-known to become, under $SO(4) \rightarrow SO(5)$, a generator of $\pi_3(SO(5)) \cong \pi_3(O) = \mathbf{Z}$.

3.3. The proof of Theorem B becomes very simple if ϕ and ψ are turned into ring homomorphisms $E_*^U \rightarrow \pi_*(U) = \bigoplus_{-1}^{\infty} \pi_s(U)$ ($\pi_{-1} = \mathbf{Z}$ generated by the ring unit) and $E_*^O \rightarrow \pi_*(O)$. For this purpose we have to define a product in $\pi_*(U)$ and $\pi_*(O)$, graded by $s+1$ for π_s . This is done by extending the product introduced in 2.2 from linear maps $f: S^s \rightarrow U$ or O to arbitrary continuous maps.

Given a continuous map $f: S^s \rightarrow U$ via $U(n)$,

$$S^s = \{x = (x_0, x_1, \dots, x_s) \in \mathbf{R}^{s+1} \text{ with } |x| = 1\},$$

we extend it to $f_0: \mathbf{R}^{s+1} \rightarrow M_n(\mathbf{C})$ by $f_0(x) = |x| f\left(\frac{x}{|x|}\right)$, $f_0(0) = 0$.

Similarly for $g: S^t \rightarrow U$ via $U(m)$, $S^t = \{y \in \mathbf{R}^{t+1} \text{ with } |y| = 1\}$. Then

$$F(x, y) = \begin{pmatrix} f_0(x) \otimes E_m & E_n \otimes g_0(y) \\ -E_n \otimes \overline{g_0(y)}^T & \overline{f_0(x)}^T \otimes E_m \end{pmatrix}$$

is a unitary $2nm \times 2nm$ matrix for all $(x, y) \in \mathbf{R}^{s+t+2}$ with $|x|^2 + |y|^2 = 1$ and thus defines a map $F: S^{s+t+1} \rightarrow U$ via $U(2nm)$. Homotopic maps f , or g respectively, yield homotopic F and we obtain a product $F = f \cup g$

$$\pi_s(U) \times \pi_t(U) \xrightarrow{\cup} \pi_{s+t+1}(U).$$

From the description of homotopy group addition in $\pi_s(U)$ as given above in 3.1 one easily checks that $f \cup g$ is distributive. Thus $\pi_*(U)$ is a ring, and so is $\pi_*(O)$, graded by $s+1$ for $\pi_s(U)$ or $\pi_s(O)$.

3.4. Bott periodicity is usually expressed in terms of complex and real K -theory. We thus use the isomorphisms

$$\pi_s(U) \cong \tilde{K}_{\mathbf{C}}(S^{s+1}) \quad \text{and} \quad \pi_s(O) \cong \tilde{K}_{\mathbf{R}}(S^{s+1}).$$

We recall that $\pi_s(U) \cong \tilde{K}_{\mathbf{C}}(S^{s+1})$ is obtained through $\pi_s(U) \cong K_{\mathbf{C}}(B^{s+1}, S^s)$ where B^{s+1} is the unit ball $\{x \in \mathbf{R}^{s+1}, |x| \leq 1\}$; the element corresponding to $f \in \pi_s(U)$ is given by two (trivial) \mathbf{C} -vector bundles over B^{s+1} , identified on S^s by means of f . It will not come as a surprise that $f \cup g$ above corresponds to the \cup -product

$$K_{\mathbf{C}}(B^{s+1}, S^s) \times K_{\mathbf{C}}(B^{t+1}, S^t) \rightarrow K_{\mathbf{C}}(B^{s+t+2}, S^{s+t+1})$$

given by the external tensor product of bundles. Indeed the map $f \cup g = F: S^{s+t+1} \rightarrow U$ via $U(2nm)$ can be interpreted as follows: One decomposes $S^{s+t+1} \subset \mathbf{R}^{s+t+2}$ (coordinates $x_0, x_1, \dots, x_s, y_0, y_1, \dots, y_t$ with $|x|^2 + |y|^2 = 1$) into $\{|x|^2 \leq \frac{1}{2}, |y|^2 \geq \frac{1}{2}\}$ homeomorphic to $B^{s+1} \times S^t$ and $\{|x|^2 \geq \frac{1}{2}, |y|^2 \leq \frac{1}{2}\}$ homeomorphic to $S^s \times B^{t+1}$; the map F is

$$\begin{pmatrix} f(x) \otimes E_m & 0 \\ 0 & \overline{f(x)}^T \otimes E_m \end{pmatrix} \quad \text{on} \quad S^s \times (0), \text{ i.e. } y = 0, |x| = 1,$$

$$\begin{pmatrix} 0 & E_n \otimes g(y) \\ -E_n \otimes \overline{g(y)}^T & 0 \end{pmatrix} \quad \text{on} \quad (0) \times S^t, \text{ i.e. } x = 0, |y| = 1.$$

Under $K_{\mathbf{C}}(B^{s+1}, S^s) \cong \tilde{K}_{\mathbf{C}}(S^{s+1})$ one then has a graded ring structure in $\bigoplus_{-1}^{\infty} \tilde{K}_{\mathbf{C}}(S^{s+1})$ isomorphic to $\pi_*(U)$. According to the Bott periodicity theorem (see [K], p. 123) this ring is the polynomial ring $\mathbf{Z}[a]$ generated by the generator of $\tilde{K}_{\mathbf{C}}(S^2)$; i.e., $\pi_*(U)$ is the polynomial ring generated by the generator a of $\pi_1(U)$.

Similarly, $\pi_*(O)$ is the commutative ring with generators $b_0 \in \pi_0(O)$, $b_3 \in \pi_3(O)$, $b_7 \in \pi_7(O)$ with relations $2b_0 = 0$, $b_0^3 = 0$, $b_3^2 = 4b_7$ ([K], p. 156-157).

To prove Theorem B we therefore only have to show:

Case U . $\rho_1 \in E_1^U$ is mapped by ϕ to $a \in \pi_1(U)$.

Case O . $\sigma_0 \in E_0^O$ is mapped by ψ to $b_0 \in \pi_0(O)$ and $\sigma_3 \in E_3^O$ to $b_3 \in \pi_3(O)$.

This has already been done in 3.2.