3. The homotopy groups of U and 0

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THEOREM 2.3. *E_*^U is the polynomial ring* $\mathbf{Z}[\rho_1]$.

2.4. THE RING E_*^O.

We denote by σ_s the generator of E_s^O ($= 0$ if $s \equiv 2, 4, 5, 6$ modulo 8; determined up to sign if $s \equiv 3, 7$ modulo 8 where $E_s^O = \mathbf{Z}$).

The generator ρ_7 ($= \rho_1^4$) $\in E_7^U$ can be given by a real ε-representation of degree 8 which we can use as generator $\sigma_7 \in E_7^O$. The ring homomorphism $\Phi: E_*^O \to E_*^U$ induced by the embedding $O \to U$, $\Phi(\sigma_7) = \rho_7$, is thus an isomorphism $E_7^O \cong E_7^U$. In E_*^O the degree of $\sigma_7\sigma_s \in E_{s+8}^O$ is $16d_s^O = d_{s+8}^O$. Hence $\sigma_7\sigma_s$ is irreducible, i.e., $= \pm\, \sigma_{s+8}$ for all s. In particular we can choose $\sigma_{15} = \sigma_7^2$, $\sigma_{23} = \sigma_7^3, \ldots, \sigma_{8r-1} = \sigma_7^r$.

PROPOSITION 2.4. *The isomorphism $E_s^O \cong E_{s+8}^O$ can be given by the product with $\sigma_7 \in E_7^O$.*

PROPOSITION 2.5. *$\sigma_7 \in E_7^O$ generates a subring of E_*^O which is the polynomial ring* $\mathbf{Z}[\sigma_7]$.

We further note that $\sigma_3 \in E_3^O$ is mapped by Φ to $2\rho_3 \in E_3^U$. From $\Phi(\sigma_3^2) = 4\rho_3^2 = 4\rho_7 = \Phi(4\sigma_7)$ we infer that $\sigma_3^2 = 4\sigma_7$. As for $\sigma_0 \in E_0^O$, it is of degree 1 and order 2, and $\sigma_0^2 \in E_1^O$ is of degree 2 and order 2, i.e., $\sigma_0^2 = \sigma_1$. Of course $\sigma_0^3 = 0$.

In summary:

THEOREM 2.6. *E_*^O is the commutative ring, graded by $s + 1$ for E_s^O, generated by $\sigma_0, \sigma_3, \sigma_7$ with the only relations $2\sigma_0 = 0$, $\sigma_0^3 = 0$, $\sigma_3^2 = 4\sigma_7$.*

3. THE HOMOTOPY GROUPS OF U AND O

3.1. We will deal explicitly with the unitary case. The orthogonal case can be treated in almost exactly the same way; any additional arguments will be mentioned wherever necessary.

In the Introduction 0.1 we associated with a set of s unitary $n \times n$ HR-matrices, i.e., with an ε-representation of G_s, a map $f: S^s \to U$ of the s-sphere $S^s \subset \mathbf{R}^{s+1}$ into the infinite unitary group U via $U(n)$. Since conjugation is homotopic to the identity, equivalent representations yield homotopic maps f (in the orthogonal case, we have to observe that conjugation can be made with a matrix from the identity component). The map $\phi: D_s^U \to \pi_s(U)$ thus obtained is a homomorphism; indeed, homotopy group addition of f and f' in $\pi_s(U(n))$ can be replaced by multiplication in

$U(n)$; this is homotopic in $U(2n)$ to the map $\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}$, and on the other hand addition in D_s^U is defined through the direct sum of representations.

If the ε-representation is restricted from D_{s+1}^U, i.e., if the set of HR-matrices belongs to a set of $s+1$ HR-matrices, f extends to a map $S^{s+1} \to U$ and is thus nullhomotopic. The homomorphism ϕ therefore induces a homomorphism $E_s^U \to \pi_s(U)$, again written ϕ. The analogue $E_s^O \to \pi_s(O)$ will be denoted by ψ. The groups E_s^U and E_s^O are 0 or cyclic generated by irreducible ε-representations, i.e., by HR-matrices of minimal size. Our claim, Theorem A, can therefore be reformulated as follows.

THEOREM B. *The homomorphisms* $\phi: E_s^U \to \pi_s(U)$ *and* $\psi: E_s^O \to \pi_s(O)$ *are isomorphisms,* $s = 0, 1, 2, \dots$.

3.2. For small values of s the claim is easily checked.

Case U

$s = 1$: E_1^U can be generated by one HR-matrix $A_1 = (i)$. Thus

$$f(x_0, x_1) = (x_0 + ix_1) \in U(1)$$

if $x_0^2 + x_1^2 = 1$. This is a generator of $\pi_1(U(1)) \cong \pi_1(U) = \mathbf{Z}$.

$s = 3$: E_3^U is generated by 3 HR-matrices

$$A_1 = \begin{pmatrix} i & \\ & -i \end{pmatrix}, \quad A_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad A_3 = \begin{pmatrix} & i \\ i & \end{pmatrix}.$$

Thus

$$f(x_0, x_1, x_2, x_3) = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \in SU(2)$$

if $\sum_0^3 x_j^2 = 1$. This is a generator of $\pi_3(SU(2))\,[=\pi_3(S^3)] \cong \pi_3(U) = \mathbf{Z}$.

Case O

$s = 0$: Empty set of HR-matrices, $f(x_0) = (x_0) \in O(1)$ if $x_0^2 = 1$, $x_0 = \pm 1$. This is a generator of $\pi_0(O(1)) \cong \pi_0(O) = \mathbf{Z}/2$.

$s = 1$: E_1^O is generated by one HR-matrix $A_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. Thus

$$f(x_0, x_1) = \begin{pmatrix} x_0 & x_1 \\ -x_1 & x_0 \end{pmatrix} \in SO(2)$$

if $x_0^2 + x_1^2 = 1$. This is a generator of $\pi_1(SO(2)) = \mathbf{Z}$; as a map $S^1 \to SO(3)$ it is a generator of $\pi_1(SO(3)) \cong \pi_1(O) = \mathbf{Z}/2$.

$s = 3$: E_3^O is generated by three 4×4 HR-matrices which yield

$$f(x_0, x_1, x_2, x_3) = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_1 & x_0 & x_3 & -x_2 \\ -x_2 & -x_3 & x_0 & x_1 \\ -x_3 & x_2 & -x_1 & x_0 \end{pmatrix} \in SO(4)$$

if $\sum_0^3 x_j^2 = 1$. This is a map $S^3 \to SO(4)$ which is well-known to become, under $SO(4) \to SO(5)$, a generator of $\pi_3(SO(5)) \cong \pi_3(O) = \mathbf{Z}$.

3.3. The proof of Theorem B becomes very simple if ϕ and ψ are turned into ring homomorphisms $E_*^U \to \pi_*(U) = \bigoplus_{-1}^{\infty} \pi_s(U)$ ($\pi_{-1} = \mathbf{Z}$ generated by the ring unit) and $E_*^O \to \pi_*(O)$. For this purpose we have to define a product in $\pi_*(U)$ and $\pi_*(O)$, graded by $s + 1$ for π_s. This is done by extending the product introduced in 2.2 from linear maps $f: S^s \to U$ or O to arbitrary continuous maps.

Given a continuous map $f: S^s \to U$ via $U(n)$,

$$S^s = \{x = (x_0, x_1, ..., x_s) \in \mathbf{R}^{s+1} \quad \text{with} \quad |x| = 1\},$$

we extend it to $f_0: \mathbf{R}^{s+1} \to M_n(\mathbf{C})$ by $f_0(x) = |x| f\left(\frac{x}{|x|}\right)$, $f_0(0) = 0$. Similarly for $g: S^t \to U$ via $U(m)$, $S^t = \{y \in \mathbf{R}^{t+1}$ with $|y| = 1\}$. Then

$$F(x, y) = \begin{pmatrix} f_0(x) \otimes E_m & E_n \otimes g_0(y) \\ -E_n \otimes \overline{g_0(y)}^T & \overline{f_0(x)}^T \otimes E_m \end{pmatrix}$$

is a unitary $2nm \times 2nm$ matrix for all $(x, y) \in \mathbf{R}^{s+t+2}$ with $|x|^2 + |y|^2 = 1$ and thus defines a map $F: S^{s+t+1} \to U$ via $U(2nm)$. Homotopic maps f, or g respectively, yield homotopic F and we obtain a product $F = f \cup g$

$$\pi_s(U) \times \pi_t(U) \overset{\cup}{\to} \pi_{s+t+1}(U).$$

From the description of homotopy group addition in $\pi_s(U)$ as given above in 3.1 one easily checks that $f \cup g$ is distributive. Thus $\pi_*(U)$ is a ring, and so is $\pi_*(O)$, graded by $s + 1$ for $\pi_s(U)$ or $\pi_s(O)$.

3.4. Bott periodicity is usually expressed in terms of complex and real K-theory. We thus use the isomorphisms

$$\pi_s(U) \cong \tilde{K}_{\mathbf{C}}(S^{s+1}) \quad \text{and} \quad \pi_s(O) \cong \tilde{K}_{\mathbf{R}}(S^{s+1}).$$

We recall that $\pi_s(U) \cong \tilde{K}_{\mathbf{C}}(S^{s+1})$ is obtained through $\pi_s(U) \cong K_{\mathbf{C}}(B^{s+1}, S^s$ where B^{s+1} is the unit ball $\{x \in \mathbf{R}^{s+1}, |x| \leqslant 1\}$; the element corresponding to $f \in \pi_s(U)$ is given by two (trivial) **C**-vector bundles over B^{s+1}, identified on S^s by means of f. It will not come as a surprise that $f \cup g$ above corresponds to the $\cup$-product

$$K_{\mathbf{C}}(B^{s+1}, S^s) \times K_{\mathbf{C}}(B^{t+1}, S^t) \to K_{\mathbf{C}}(B^{s+t+2}, S^{s+t+1})$$

given by the external tensor product of bundles. Indeed the map $f \cup g = F: S^{s+t+1} \to U$ via $U(2nm)$ can be interpreted as follows: One decomposes $S^{s+t+1} \subset \mathbf{R}^{s+t+2}$ (coordinates $x_0, x_1, ..., x_s, y_0, y_1, ..., y_t$ with $|x|^2 + |y|^2 = 1$) into $\{|x|^2 \leqslant \frac{1}{2}, |y|^2 \geqslant \frac{1}{2}\}$ homeomorphic to $B^{s+1} \times S^t$ and $\{|x|^2 \geqslant \frac{1}{2}, |y|^2 \leqslant \frac{1}{2}\}$ homeomorphic to $S^s \times B^{t+1}$; the map F is

$$\begin{pmatrix} f(x) \otimes E_m & 0 \\ 0 & \overline{f(x)}^T \otimes E_m \end{pmatrix} \quad \text{on} \quad S^s \times (0), \text{ i.e. } y = 0, |x| = 1,$$

$$\begin{pmatrix} 0 & E_n \otimes g(y) \\ -E_n \otimes \overline{g(y)}^T & 0 \end{pmatrix} \quad \text{on} \quad (0) \times S^t, \text{ i.e. } x = 0, |y| = 1.$$

Under $K_{\mathbf{C}}(B^{s+1}, S^s) \cong \tilde{K}_{\mathbf{C}}(S^{s+1})$ one then has a graded ring structure in $\bigoplus_{-1}^{\infty} \tilde{K}_{\mathbf{C}}(S^{s+1})$ isomorphic to $\pi_*(U)$. According to the Bott periodicity theorem (see [K], p. 123) this ring is the polynomial ring $\mathbf{Z}[a]$ generated by the generator of $\tilde{K}_{\mathbf{C}}(S^2)$; i.e., $\pi_*(U)$ is the polynomial ring generated by the generator a of $\pi_1(U)$.

Similarly, $\pi_*(O)$ is the commutative ring with generators $b_0 \in \pi_0(O)$, $b_3 \in \pi_3(O)$, $b_7 \in \pi_7(O)$ with relations $2b_0 = 0$, $b_0^3 = 0$, $b_3^2 = 4b_7$ ([K], p. 156-157).

To prove Theorem B we therefore only have to show:

Case U. $\rho_1 \in E_1^U$ is mapped by ϕ to $a \in \pi_1(U)$.

Case O. $\sigma_0 \in E_0^O$ is mapped by ψ to $b_0 \in \pi_0(O)$ and $\sigma_3 \in E_3^O$ to $b_3 \in \pi_3(O)$.

This has already been done in 3.2.