

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 35 (1989)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** HURWITZ-RADON MATRICES AND PERIODICITY MODULO 8  
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**Kapitel:** 3. The homotopy groups of  $U$  and  $O$   
**DOI:** <https://doi.org/10.5169/seals-57365>

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THEOREM 2.3.  $E_*^U$  is the polynomial ring  $\mathbb{Z}[\rho_1]$ .

#### 2.4. THE RING $E_*^O$ .

We denote by  $\sigma_s$  the generator of  $E_s^O$  ( $= 0$  if  $s \equiv 2, 4, 5, 6$  modulo 8; determined up to sign if  $s \equiv 3, 7$  modulo 8 where  $E_s^O = \mathbb{Z}$ ).

The generator  $\rho_7 (= \rho_1^4) \in E_7^U$  can be given by a real  $\varepsilon$ -representation of degree 8 which we can use as generator  $\sigma_7 \in E_7^O$ . The ring homomorphism  $\Phi: E_*^O \rightarrow E_*^U$  induced by the embedding  $O \rightarrow U$ ,  $\Phi(\sigma_7) = \rho_7$ , is thus an isomorphism  $E_7^O \cong E_7^U$ . In  $E_*^O$  the degree of  $\sigma_7 \sigma_s \in E_{s+8}^O$  is  $16d_s^O = d_{s+8}^O$ . Hence  $\sigma_7 \sigma_s$  is irreducible, i.e.,  $= \pm \sigma_{s+8}$  for all  $s$ . In particular we can choose  $\sigma_{15} = \sigma_7^2$ ,  $\sigma_{23} = \sigma_7^3$ , ...,  $\sigma_{8r-1} = \sigma_7^r$ .

PROPOSITION 2.4. The isomorphism  $E_s^O \cong E_{s+8}^O$  can be given by the product with  $\sigma_7 \in E_7^O$ .

PROPOSITION 2.5.  $\sigma_7 \in E_7^O$  generates a subring of  $E_*^O$  which is the polynomial ring  $\mathbb{Z}[\sigma_7]$ .

We further note that  $\sigma_3 \in E_3^O$  is mapped by  $\Phi$  to  $2\rho_3 \in E_3^U$ . From  $\Phi(\sigma_3^2) = 4\rho_3^2 = 4\rho_7 = \Phi(4\sigma_7)$  we infer that  $\sigma_3^2 = 4\sigma_7$ . As for  $\sigma_0 \in E_0^O$ , it is of degree 1 and order 2, and  $\sigma_0^2 \in E_1^O$  is of degree 2 and order 2, i.e.,  $\sigma_0^2 = \sigma_1$ . Of course  $\sigma_0^3 = 0$ .

In summary:

THEOREM 2.6.  $E_*^O$  is the commutative ring, graded by  $s+1$  for  $E_s^O$ , generated by  $\sigma_0, \sigma_3, \sigma_7$  with the only relations  $2\sigma_0 = 0$ ,  $\sigma_0^3 = 0$ ,  $\sigma_3^2 = 4\sigma_7$ .

### 3. THE HOMOTOPY GROUPS OF $U$ AND $O$

3.1. We will deal explicitly with the unitary case. The orthogonal case can be treated in almost exactly the same way; any additional arguments will be mentioned wherever necessary.

In the Introduction 0.1 we associated with a set of  $s$  unitary  $n \times n$  HR-matrices, i.e., with an  $\varepsilon$ -representation of  $G_s$ , a map  $f: S^s \rightarrow U$  of the  $s$ -sphere  $S^s \subset \mathbb{R}^{s+1}$  into the infinite unitary group  $U$  via  $U(n)$ . Since conjugation is homotopic to the identity, equivalent representations yield homotopic maps  $f$  (in the orthogonal case, we have to observe that conjugation can be made with a matrix from the identity component). The map  $\phi: D_s^U \rightarrow \pi_s(U)$  thus obtained is a homomorphism; indeed, homotopy group addition of  $f$  and  $f'$  in  $\pi_s(U(n))$  can be replaced by multiplication in

$U(n)$ ; this is homotopic in  $U(2n)$  to the map  $\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}$ , and on the other hand addition in  $D_s^U$  is defined through the direct sum of representations.

If the  $\varepsilon$ -representation is restricted from  $D_{s+1}^U$ , i.e., if the set of HR-matrices belongs to a set of  $s+1$  HR-matrices,  $f$  extends to a map  $S^{s+1} \rightarrow U$  and is thus nullhomotopic. The homomorphism  $\phi$  therefore induces a homomorphism  $E_s^U \rightarrow \pi_s(U)$ , again written  $\phi$ . The analogue  $E_s^O \rightarrow \pi_s(O)$  will be denoted by  $\psi$ . The groups  $E_s^U$  and  $E_s^O$  are 0 or cyclic generated by irreducible  $\varepsilon$ -representations, i.e., by HR-matrices of minimal size. Our claim, Theorem A, can therefore be reformulated as follows.

**THEOREM B.** *The homomorphisms  $\phi: E_s^U \rightarrow \pi_s(U)$  and  $\psi: E_s^O \rightarrow \pi_s(O)$  are isomorphisms,  $s = 0, 1, 2, \dots$ .*

3.2. For small values of  $s$  the claim is easily checked.

*Case U*

$s = 1$ :  $E_1^U$  can be generated by one HR-matrix  $A_1 = (i)$ . Thus

$$f(x_0, x_1) = (x_0 + ix_1) \in U(1)$$

if  $x_0^2 + x_1^2 = 1$ . This is a generator of  $\pi_1(U(1)) \cong \pi_1(U) = \mathbf{Z}$ .

$s = 3$ :  $E_3^U$  is generated by 3 HR-matrices

$$A_1 = \begin{pmatrix} i & \\ & -i \end{pmatrix}, A_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, A_3 = \begin{pmatrix} & i \\ i & \end{pmatrix}.$$

Thus

$$f(x_0, x_1, x_2, x_3) = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \in SU(2)$$

if  $\sum_{j=0}^3 x_j^2 = 1$ . This is a generator of  $\pi_3(SU(2)) [= \pi_3(S^3)] \cong \pi_3(U) = \mathbf{Z}$ .

*Case O*

$s = 0$ : Empty set of HR-matrices,  $f(x_0) = (x_0) \in O(1)$  if  $x_0^2 = 1$ ,  $x_0 = \pm 1$ . This is a generator of  $\pi_0(O(1)) \cong \pi_0(O) = \mathbf{Z}/2$ .

$s = 1$ :  $E_1^O$  is generated by one HR-matrix  $A_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ . Thus

$$f(x_0, x_1) = \begin{pmatrix} x_0 & x_1 \\ -x_1 & x_0 \end{pmatrix} \in SO(2)$$

if  $x_0^2 + x_1^2 = 1$ . This is a generator of  $\pi_1(SO(2)) = \mathbf{Z}$ ; as a map  $S^1 \rightarrow SO(3)$  it is a generator of  $\pi_1(SO(3)) \cong \pi_1(O) = \mathbf{Z}/2$ .

$s = 3$ :  $E_3^O$  is generated by three  $4 \times 4$  HR-matrices which yield

$$f(x_0, x_1, x_2, x_3) = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_1 & x_0 & x_3 & -x_2 \\ -x_2 & -x_3 & x_0 & x_1 \\ -x_3 & x_2 & -x_1 & x_0 \end{pmatrix} \in SO(4)$$

if  $\sum_{j=0}^3 x_j^2 = 1$ . This is a map  $S^3 \rightarrow SO(4)$  which is well-known to become, under  $SO(4) \rightarrow SO(5)$ , a generator of  $\pi_3(SO(5)) \cong \pi_3(O) = \mathbf{Z}$ .

3.3. The proof of Theorem B becomes very simple if  $\phi$  and  $\psi$  are turned into ring homomorphisms  $E_*^U \rightarrow \pi_*(U) = \bigoplus_{s=-1}^{\infty} \pi_s(U)$  ( $\pi_{-1} = \mathbf{Z}$  generated by the ring unit) and  $E_*^O \rightarrow \pi_*(O)$ . For this purpose we have to define a product in  $\pi_*(U)$  and  $\pi_*(O)$ , graded by  $s+1$  for  $\pi_s$ . This is done by extending the product introduced in 2.2 from linear maps  $f: S^s \rightarrow U$  or  $O$  to arbitrary continuous maps.

Given a continuous map  $f: S^s \rightarrow U$  via  $U(n)$ ,

$$S^s = \{x = (x_0, x_1, \dots, x_s) \in \mathbf{R}^{s+1} \text{ with } |x| = 1\},$$

we extend it to  $f_0: \mathbf{R}^{s+1} \rightarrow M_n(\mathbf{C})$  by  $f_0(x) = |x| f\left(\frac{x}{|x|}\right)$ ,  $f_0(0) = 0$ .

Similarly for  $g: S^t \rightarrow U$  via  $U(m)$ ,  $S^t = \{y \in \mathbf{R}^{t+1} \text{ with } |y| = 1\}$ . Then

$$F(x, y) = \begin{pmatrix} f_0(x) \otimes E_m & E_n \otimes g_0(y) \\ -E_n \otimes \overline{g_0(y)}^T & \overline{f_0(x)}^T \otimes E_m \end{pmatrix}$$

is a unitary  $2nm \times 2nm$  matrix for all  $(x, y) \in \mathbf{R}^{s+t+2}$  with  $|x|^2 + |y|^2 = 1$  and thus defines a map  $F: S^{s+t+1} \rightarrow U$  via  $U(2nm)$ . Homotopic maps  $f$ , or  $g$  respectively, yield homotopic  $F$  and we obtain a product  $F = f \cup g$

$$\pi_s(U) \times \pi_t(U) \xrightarrow{\cup} \pi_{s+t+1}(U).$$

From the description of homotopy group addition in  $\pi_s(U)$  as given above in 3.1 one easily checks that  $f \cup g$  is distributive. Thus  $\pi_*(U)$  is a ring, and so is  $\pi_*(O)$ , graded by  $s+1$  for  $\pi_s(U)$  or  $\pi_s(O)$ .

3.4. Bott periodicity is usually expressed in terms of complex and real  $K$ -theory. We thus use the isomorphisms

$$\pi_s(U) \cong \tilde{K}_{\mathbf{C}}(S^{s+1}) \quad \text{and} \quad \pi_s(O) \cong \tilde{K}_{\mathbf{R}}(S^{s+1}).$$

We recall that  $\pi_s(U) \cong \tilde{K}_{\mathbf{C}}(S^{s+1})$  is obtained through  $\pi_s(U) \cong K_{\mathbf{C}}(B^{s+1}, S^s)$  where  $B^{s+1}$  is the unit ball  $\{x \in \mathbf{R}^{s+1}, |x| \leq 1\}$ ; the element corresponding to  $f \in \pi_s(U)$  is given by two (trivial)  $\mathbf{C}$ -vector bundles over  $B^{s+1}$ , identified on  $S^s$  by means of  $f$ . It will not come as a surprise that  $f \cup g$  above corresponds to the  $\cup$ -product

$$K_{\mathbf{C}}(B^{s+1}, S^s) \times K_{\mathbf{C}}(B^{t+1}, S^t) \rightarrow K_{\mathbf{C}}(B^{s+t+2}, S^{s+t+1})$$

given by the external tensor product of bundles. Indeed the map  $f \cup g = F: S^{s+t+1} \rightarrow U$  via  $U(2nm)$  can be interpreted as follows: One decomposes  $S^{s+t+1} \subset \mathbf{R}^{s+t+2}$  (coordinates  $x_0, x_1, \dots, x_s, y_0, y_1, \dots, y_t$  with  $|x|^2 + |y|^2 = 1$ ) into  $\{|x|^2 \leq \frac{1}{2}, |y|^2 \geq \frac{1}{2}\}$  homeomorphic to  $B^{s+1} \times S^t$  and  $\{|x|^2 \geq \frac{1}{2}, |y|^2 \leq \frac{1}{2}\}$  homeomorphic to  $S^s \times B^{t+1}$ ; the map  $F$  is

$$\begin{pmatrix} f(x) \otimes E_m & 0 \\ 0 & \overline{f(x)}^T \otimes E_m \end{pmatrix} \quad \text{on} \quad S^s \times (0), \text{ i.e. } y = 0, |x| = 1,$$

$$\begin{pmatrix} 0 & E_n \otimes g(y) \\ -E_n \otimes \overline{g(y)}^T & 0 \end{pmatrix} \quad \text{on} \quad (0) \times S^t, \text{ i.e. } x = 0, |y| = 1.$$

Under  $K_{\mathbf{C}}(B^{s+1}, S^s) \cong \tilde{K}_{\mathbf{C}}(S^{s+1})$  one then has a graded ring structure in  $\bigoplus_{-1}^{\infty} \tilde{K}_{\mathbf{C}}(S^{s+1})$  isomorphic to  $\pi_*(U)$ . According to the Bott periodicity theorem (see [K], p. 123) this ring is the polynomial ring  $\mathbf{Z}[a]$  generated by the generator of  $\tilde{K}_{\mathbf{C}}(S^2)$ ; i.e.,  $\pi_*(U)$  is the polynomial ring generated by the generator  $a$  of  $\pi_1(U)$ .

Similarly,  $\pi_*(O)$  is the commutative ring with generators  $b_0 \in \pi_0(O)$ ,  $b_3 \in \pi_3(O)$ ,  $b_7 \in \pi_7(O)$  with relations  $2b_0 = 0$ ,  $b_0^3 = 0$ ,  $b_3^2 = 4b_7$  ([K], p. 156-157).

To prove Theorem B we therefore only have to show:

Case  $U$ .  $\rho_1 \in E_1^U$  is mapped by  $\phi$  to  $a \in \pi_1(U)$ .

Case  $O$ .  $\sigma_0 \in E_0^O$  is mapped by  $\psi$  to  $b_0 \in \pi_0(O)$  and  $\sigma_3 \in E_3^O$  to  $b_3 \in \pi_3(O)$ .

This has already been done in 3.2.