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The values of  $d_s^O$  follow immediately from the  $I_s$  and the  $d_s^U$ . The values  $n_0$  for the case  $O$ , as given in the Introduction, are the  $d_s^O$ .

## 2. THE REDUCED $\varepsilon$ -REPRESENTATION RING

2.1. For all  $s \geq 0$  the group  $G_s$  is the subgroup of  $G_{s+1}$  obtained by omitting the generator  $a_{s+1}$ ; let  $h_s: G_s \rightarrow G_{s+1}$  be the embedding homomorphism. Via  $h_s$  we can restrict an  $\varepsilon$ -representation of  $G_{s+1}$  to  $G_s$ , which in terms of HR-matrices means omitting  $A_{s+1}$ .

Let  $h_s^*: D_{s+1}^U \rightarrow D_s^U$  be the corresponding homomorphism of Grothendieck groups, and  $E_s^U = D_s^U / h_s^* D_{s+1}^U$  the "reduced" groups; similarly  $E_s^O = D_s^O / h_s^* D_{s+1}^O$ . They can easily be computed by means of the characters of  $\varepsilon$ -representations, as follows.

For  $Q$  and  $D$  the character of an irreducible unitary  $\varepsilon$ -representation is 0 except on 1 and  $\varepsilon$ . For  $C$  and  $K$  it is  $\neq 0$  on all 4 elements; on the essential generator ( $\neq \varepsilon$ ) of  $C$  it is  $+i$  or  $-i$  for the two inequivalent representations, and  $+1$  or  $-1$  in the case of  $K$ . For  $G_s$ ,  $s$  even, we infer from the table (2) that the character is 0 except on 1,  $\varepsilon$ . For  $G_s$ ,  $s$  odd, the character is 0 except on 1,  $\varepsilon$  and two further elements  $z, \varepsilon z$ ; on these the two inequivalent  $\varepsilon$ -representations differ just by the sign of the character.

If  $s$  is even,  $d_{s+1}^U = d_s^U = 2^{s/2}$ ; thus the restriction of an irreducible  $\varepsilon$ -representation must be irreducible, whence  $h_s^* D_{s+1}^U = D_s^U$ ,  $E_s^U = 0$ . If  $s$  is odd,  $d_{s+1}^U = 2d_s^U = 2^{(s+1)/2}$ ; thus the restriction is the sum of two irreducible  $\varepsilon$ -representations, and since the character is 0 (except on 1,  $\varepsilon$ ) these two must be inequivalent. Therefore  $h_s^* D_{s+1}^U$  is the "diagonal" of  $D_s^U = \mathbf{Z} \oplus \mathbf{Z}$ , and  $E_s^U = \mathbf{Z}$ ; its generator  $\rho_s$  is represented by either of the two inequivalent irreducible  $\varepsilon$ -representations of  $G_s$ ,  $-\rho_s$  by the other one.

In the orthogonal case the  $E_s^O$  are computed similarly from (3). Since  $d_1^O = 2$  and  $d_0^O = 1$ , the restriction from  $D_1^O$  to  $D_0^O$  yields twice the generator, and  $E_0^O = \mathbf{Z}/2$ ; the same argument holds for  $s \equiv 0 \pmod 8$ ,  $d_{s+1}^O = 2d_s^O$ . Since  $d_2^O = 4$  and  $d_1^O = 2$ , we get  $E_1^O = \mathbf{Z}/2$ . From  $d_3^O = d_2^O = 4$  we get  $E_2^O = 0$ . As for  $s = 3$ , the character argument shows that  $h_3^* D_4^O = \text{diagonal of } D_3^O (= \mathbf{Z} \oplus \mathbf{Z})$ , and  $E_3^O = \mathbf{Z}$ . For  $s = 4, 5, 6$  the dimensions  $d_{s+1}^O = d_s^O$  show that  $E_4^O = E_5^O = E_6^O = 0$ . For  $s = 7$ , the character argument yields  $h_7^* D_8^O = \text{diagonal of } D_7^O (= \mathbf{Z} \oplus \mathbf{Z})$ , and  $E_7^O = \mathbf{Z}$ . Finally one has, for all  $s$ ,  $E_{s+8}^O \cong E_s^O$ .

These results are summarized in the table

(4)	$s$	0	1	2	3	4	5	6	7	8	9	...
	$E_s^U$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	
	$E_s^O$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	

According to the Bott periodicity theorems the above table is just that of the  $\pi_s(U)$  and  $\pi_s(O)$ ,  $s = 0, 1, 2, \dots$ . Before studying the relation as stated in Theorem A we establish product structures in the reduced Grothendieck groups of  $\varepsilon$ -representations, i.e., of HR-matrices.

2.2. We consider HR-matrices  $A_1, A_2, \dots, A_s \in U(n)$  and put, for

$$x = (x_0, x_1, \dots, x_s) \in \mathbf{R}^{s+1}$$

and  $A_0 = E_n$  ( $n \times n$  unit matrix)

$$f(x) = \sum_0^s x_j A_j.$$

For all  $x$  with  $|x| = 1$ ,  $f(x)$  is a unitary matrix: this is, as mentioned in the Introduction, precisely the meaning of the HR-matrix relations (1).

Let further  $B_1, B_2, \dots, B_t \in U(m)$  be HR-matrices, and for

$$y = (y_0, y_1, \dots, y_t) \in \mathbf{R}^{t+1}, \quad B_0 = E_m,$$

$$g(y) = \sum_0^t y_k B_k;$$

$g(y) \in U(m)$  for all  $y$  with  $|y| = 1$ . We define  $F$  by

$$F(x, y) = \begin{pmatrix} f(x) \otimes E_m & E_n \otimes g(y) \\ -E_n \otimes \overline{g(y)}^T & \overline{f(x)}^T \otimes E_m \end{pmatrix}.$$

One immediately checks that  $F(x, y) \bar{F}^T(x, y) = (|x|^2 + |y|^2) E_{2nm}$ . Thus  $F(x, y) \in U(2nm)$  for all  $(x, y) \in \mathbf{R}^{s+t+2}$  with  $|x|^2 + |y|^2 = 1$ . Since the coefficient matrix of  $x_0$  is  $E_{2nm}$  the coefficient matrices of  $x_1, \dots, x_s, y_0, \dots, y_t$  constitute a set of  $s + t + 1$  HR-matrices  $\in U(2nm)$ . They are, explicitly,

$$(5) \quad \begin{pmatrix} A_j \otimes E_m & 0 \\ 0 & -A_j \otimes E_m \end{pmatrix}, \begin{pmatrix} 0 & E_{nm} \\ -E_{nm} & 0 \end{pmatrix}, \begin{pmatrix} 0 & E_n \otimes B_k \\ E_n \otimes B_k & 0 \end{pmatrix}$$

with  $j = 1, \dots, s$  and  $k = 1, \dots, t$ . In other words, we have a product of  $\varepsilon$ -representations of  $G_s$  and  $G_t$

$$D_s^U \times D_t^U \xrightarrow{\cup} D_{s+t+1}^U.$$

Since addition in  $D_s^U$  is by the direct sum of  $\varepsilon$ -representations this product is clearly distributive. Associativity (up to equivalence) is easily checked. We thus get a ring structure in  $D_*^U = \bigoplus_{-1}^{\infty} D_s^U$ ; we have added the term  $D_{-1}^U = \mathbf{Z}$  generated by the ring unit. The ring  $D_*^U$  is graded if the grading is by  $s + 1$  for  $D_s$ .

From the HR-matrices (5) of the product one notes that if one of the two factors is restricted from  $D_*^U$  so is the product; i.e.,  $h*D_*^U$  is a (graded) ideal in  $D_*^U$ , and we get a (graded) ring structure in  $D_*^U/h*D_*^U = E_*^U$ .

The same procedure yields, of course, a (graded) ring structure in  $E_*^O = \bigoplus_{s=-1}^{\infty} E_s^O$ , with grading  $s + 1$  for  $E_s^O$ . In 2.3 and 2.4 below these rings are described explicitly.

*Remark 2.1.* An easy computation shows that the rings  $E_*^U$  and  $E_*^O$  are anticommutative with respect to the grading, i.e., commutative except for the factor  $(-1)^{(s+1)(t+1)}$ . This will not really be used since the  $E_s^U$  and  $E_s^O$  are all 0,  $\mathbf{Z}$  or  $\mathbf{Z}/2$ . We just note that in the case  $\mathbf{Z}$ , with generator  $\rho_s$ ,  $-\rho_s$  is given by the other equivalence class of irreducible  $\varepsilon$ -representations, see 2.1.

### 2.3. The ring $E_*^U$ .

The generator  $\rho_s$  of  $E_s^U$ , given by an irreducible unitary  $\varepsilon$ -representation of  $G_s$ , has degree  $2^{s/2}$  if  $s$  is even,  $2^{(s-1)/2}$  if  $s$  is odd. The product  $\rho_s \rho_t \in E_{s+t+1}^U$  has degree

$$\begin{array}{ll} 2^{(s+t+2)/2} & \text{if } s \text{ and } t \text{ are even,} \\ 2^{(s+t+1)/2} & \text{if } s \text{ is even, } t \text{ odd, or vice-versa,} \\ 2^{(s+t)/2} & \text{if } s \text{ and } t \text{ are odd.} \end{array}$$

Thus, unless both  $s$  and  $t$  are even, the product is irreducible, i.e.,  $\rho_s \rho_t = \pm \rho_{s+t+1}$ . After choice of  $\rho_1 \in E_1^U$  we can choose  $\rho_3 = \rho_1^2$ ,  $\rho_5 = \rho_1 \rho_3 = \rho_3 \rho_1 = \rho_1^3$ , ..., and for all odd  $s = 2r - 1$ ,  $\rho_s = \rho_1^r$ ; for even  $s$ ,  $E_s^U = 0$ .

**PROPOSITION 2.2.** *The product with  $\rho_1 \in E_1^U$  is an isomorphism  $E_s^U \cong E_{s+2}^U$  for all  $s$ . For odd  $s = 2l - 1$  we choose*

$$\rho_{2l-1} = \rho_1^l, l = 1, 2, 3, \dots$$

THEOREM 2.3.  $E_*^U$  is the polynomial ring  $\mathbb{Z}[\rho_1]$ .

#### 2.4. THE RING $E_*^O$ .

We denote by  $\sigma_s$  the generator of  $E_s^O$  ( $= 0$  if  $s \equiv 2, 4, 5, 6$  modulo 8; determined up to sign if  $s \equiv 3, 7$  modulo 8 where  $E_s^O = \mathbb{Z}$ ).

The generator  $\rho_7 (= \rho_1^4) \in E_7^U$  can be given by a real  $\varepsilon$ -representation of degree 8 which we can use as generator  $\sigma_7 \in E_7^O$ . The ring homomorphism  $\Phi: E_*^O \rightarrow E_*^U$  induced by the embedding  $O \rightarrow U$ ,  $\Phi(\sigma_7) = \rho_7$ , is thus an isomorphism  $E_7^O \cong E_7^U$ . In  $E_*^O$  the degree of  $\sigma_7 \sigma_s \in E_{s+8}^O$  is  $16d_s^O = d_{s+8}^O$ . Hence  $\sigma_7 \sigma_s$  is irreducible, i.e.,  $= \pm \sigma_{s+8}$  for all  $s$ . In particular we can choose  $\sigma_{15} = \sigma_7^2$ ,  $\sigma_{23} = \sigma_7^3$ , ...,  $\sigma_{8r-1} = \sigma_7^r$ .

PROPOSITION 2.4. The isomorphism  $E_s^O \cong E_{s+8}^O$  can be given by the product with  $\sigma_7 \in E_7^O$ .

PROPOSITION 2.5.  $\sigma_7 \in E_7^O$  generates a subring of  $E_*^O$  which is the polynomial ring  $\mathbb{Z}[\sigma_7]$ .

We further note that  $\sigma_3 \in E_3^O$  is mapped by  $\Phi$  to  $2\rho_3 \in E_3^U$ . From  $\Phi(\sigma_3^2) = 4\rho_3^2 = 4\rho_7 = \Phi(4\sigma_7)$  we infer that  $\sigma_3^2 = 4\sigma_7$ . As for  $\sigma_0 \in E_0^O$ , it is of degree 1 and order 2, and  $\sigma_0^2 \in E_1^O$  is of degree 2 and order 2, i.e.,  $\sigma_0^2 = \sigma_1$ . Of course  $\sigma_0^3 = 0$ .

In summary:

THEOREM 2.6.  $E_*^O$  is the commutative ring, graded by  $s+1$  for  $E_s^O$ , generated by  $\sigma_0, \sigma_3, \sigma_7$  with the only relations  $2\sigma_0 = 0$ ,  $\sigma_0^3 = 0$ ,  $\sigma_3^2 = 4\sigma_7$ .

### 3. THE HOMOTOPY GROUPS OF $U$ AND $O$

3.1. We will deal explicitly with the unitary case. The orthogonal case can be treated in almost exactly the same way; any additional arguments will be mentioned wherever necessary.

In the Introduction 0.1 we associated with a set of  $s$  unitary  $n \times n$  HR-matrices, i.e., with an  $\varepsilon$ -representation of  $G_s$ , a map  $f: S^s \rightarrow U$  of the  $s$ -sphere  $S^s \subset \mathbb{R}^{s+1}$  into the infinite unitary group  $U$  via  $U(n)$ . Since conjugation is homotopic to the identity, equivalent representations yield homotopic maps  $f$  (in the orthogonal case, we have to observe that conjugation can be made with a matrix from the identity component). The map  $\phi: D_s^U \rightarrow \pi_s(U)$  thus obtained is a homomorphism; indeed, homotopy group addition of  $f$  and  $f'$  in  $\pi_s(U(n))$  can be replaced by multiplication in