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HURWITZ-RADON MATRICES AND PERIODICITY MODULO 8

by Beno ECKMANN

0. INTRODUCTION

0.1. We consider complex $n \times n$ - matrices A_1, A_2, \dots, A_s , either all unitary (case U) or all orthogonal (case O); they are called Hurwitz-Radon matrices, in short HR-matrices, if

$$(1) \quad A_j^2 = -E, \quad A_j A_k + A_k A_j = 0, \quad j, k = 1, 2, \dots, s, \quad j \neq k;$$

E or E_n denotes the unit matrix. Such matrices are well-known to exist, even with entries $0, \pm 1, \pm i$ (case U) or $0, \pm 1$ (case O). The possible values of n are multiples mn_0 , $m = 1, 2, 3, \dots$ where in case U , $n_0 = 2^{s/2}$ if s is even, $n_0 = 2^{(s-1)/2}$ if s is odd. In case O , $n_0 = 2^{(s-1)/2}$ if $s \equiv 7 \pmod{8}$; $n_0 = 2^{s/2}$ if $s \equiv 0, 6$; $n_0 = 2^{(s+1)/2}$ if $s \equiv 1, 3, 5$; and $n_0 = 2^{(s+2)/2}$ if $s \equiv 2, 4 \pmod{8}$.

If we put $A_0 = E$ the relations (1) are equivalent to

$$f_s(x_0, x_1, \dots, x_s) = \sum_0^s x_j A_j$$

being a unitary, or orthogonal respectively, matrix for all real x_j with $\sum_0^s x_j^2 = 1$. Thus f_s can be considered as a map $S^s \rightarrow U$ via $U(n)$, or $S^s \rightarrow O$ via $O(n)$ where $U = \varinjlim U(k)$ and $O = \varinjlim O(k)$ are the infinite unitary and orthogonal groups. We also write f_s for the homotopy class of that map, $f_s \in \pi_s(U)$ or $\pi_s(O)$. We recall that by the Bott periodicity theorems these groups are cyclic or 0.

THEOREM A. *If A_1, A_2, \dots, A_s are HR-matrices of minimal size $n = n_0(s)$ then f_s is a generator of $\pi_s(U)$, or $\pi_s(O)$ respectively, $s = 0, 1, 2, \dots$.*

Remark 1. For $s = 0$ (empty set of HR-matrices) we have $f_0(x_0) = x_0(1)$, $x_0^2 = 1$; i.e., $f_0(1) = (1)$, $f_0(-1) = (-1)$, $f_0: S^0 \rightarrow O(1) \rightarrow O$. For $s > 0$, $f_0: S^s \rightarrow O$ clearly factors through $SO(n) \rightarrow SO$ (U being connected, the analogue is irrelevant in the unitary case).

Remark 2. The problem originally solved by Hurwitz [H] and Radon [R] refers to the case O : One asks for complex bilinear forms $z = f(x, y) = (\sum_0^s x_j A_j) y$, where $z = (z_1, \dots, z_n)$, $y = (y_1, \dots, y_n)$, $x = (x_0, \dots, x_s)$, such that

$$z_1^2 + \dots + z_n^2 = (x_0^2 + \dots + x_s^2) (y_1^2 + \dots + y_n^2).$$

This means that $\sum_0^s x_j A_j$ is orthogonal, i.e. leaves invariant $\sum_0^n y_j^2$ except for the factor $\sum_0^s x_j^2$; and thus, since we may assume $A_0 = E$, that A_1, \dots, A_s is a set of orthogonal HR-matrices in the sense of (1).

The case U refers to the analogous problem for the identity

$$|z_1|^2 + \dots + |z_n|^2 = (x_0^2 + \dots + x_s^2) (|y_1|^2 + \dots + |y_n|^2)$$

where y and z are complex, and x real.

0.2. The symplectic case: It is also of interest to consider HR-matrices, i.e. matrices fulfilling (1), which are symplectic. A linear combination $\sum_0^s x_j A_j$ of $2n \times 2n$ -matrices with $A_0 = E$ is symplectic up to the factor $\sum_0^s x_j^2$ if and only if A_1, \dots, A_s is a set of symplectic HR-matrices (Proposition 4.1).

We restrict attention to unitary symplectic matrices, i.e., to the group $Sp(n) \subset U(2n)$, and write Sp for the infinite symplectic group $\varinjlim Sp(k)$. With a set A_1, \dots, A_s of unitary symplectic HR-matrices, and $A_0 = E$, we associate the map $f_s(x_0, x_1, \dots, x_s) = \sum_0^s x_j A_j$, $\sum_0^s x_j^2 = 1$, of S^s into Sp via $Sp(n)$; we also write f_s for the corresponding element of $\pi_s(Sp)$, known to be 0 or cyclic.

THEOREM A'. *If A_1, \dots, A_s are unitary symplectic HR-matrices of minimal size $2n_0$ then f_s is a generator of $\pi_s(Sp)$.*

0.3. The paper is organized as follows. We first recall (Section 1) that the HR-matrix problem can be formulated in terms of representations of certain finite group G_s , $s = 0, 1, 2, \dots$ introduced by the author [E], and discuss these representations using the elegant description of [LS]. In Section 2 the "reduced" Grothendieck groups of representations E_s^U and E_s^O are

computed; they turn out to be isomorphic to $\pi_s(U)$ and $\pi_s(O)$ respectively. Moreover a product is defined in the direct sum of the $E_s^U(E_s^O)$ turning it into a graded ring $E_*^U(E_*^O)$. The claim of Theorem A is proved in Section 3; we show that the maps $\phi: E_s^U \rightarrow \pi_s(U)$, $\psi: E_s^O \rightarrow \pi_s(O)$ given by the f_s of 0.1 are isomorphisms. Using the product structure in $\pi_*(U)$ and $\pi_*(O)$ known from K -theory the proof reduces to simple verifications in low dimensions. The symplectic case is dealt with in Section 4. In Section 5 we make a remark concerning the "linearization phenomenon" for the homotopy groups of U , O and Sp .

1. THE GROUPS G_s AND THEIR REPRESENTATIONS

1.1. We will denote throughout by G_s the group given by the presentation

$$G_s = \langle \varepsilon, a_1, \dots, a_s \mid \varepsilon^2 = 1, a_j^2 = \varepsilon, a_j a_k = \varepsilon a_k a_j, j, k = 1, 2, \dots, s, j \neq k \rangle.$$

Clearly any set A_1, \dots, A_s of HR-matrices yields a (unitary or orthogonal) representation of G_s of degree n by $\varepsilon \mapsto -E$, $a_j \mapsto A_j$, $j = 1, 2, \dots, s$. Conversely a representation of G_s with $\varepsilon \mapsto -E$, in short an ε -representation, yields a set of s HR-matrices. For the elementary properties of G_s and its representations we refer to [E]. We just recall that the order of G_s is 2^{s+1} , that ε is central, and that the irreducible unitary ε -representations of G_s are of degree $2^{s/2}$ if s is even (one equivalence class), of degree $2^{(s-1)/2}$ if s is odd (two equivalence classes). These degrees are the minimal values n_0 in case U . As for the case O , one has to recall that a representation is equivalent to an orthogonal one if and only if it is equivalent to a real (and orthogonal) one. Thus, unless an irreducible unitary ε -representation is already real, one has to add its conjugate-complex representation, and the discussion of the various cases depending on s yields the minimal values n_0 (case O) mentioned in the introduction; in other words, the degrees of the irreducible orthogonal ε -representations of G_s .

1.2. A very simple and useful scheme for studying the groups G_s and their ε -representations (and many other things) has been devised by T. Y. Lam and T. Smith [LS]. They have expressed the G_s as products of very small and well-known groups. Namely $C = G_1$, the cyclic group of order 4; $Q = G_2$, the quaternionic group of order 8; K , the Klein 4-group; and D , the dihedral group of order 8. Although K and D do not belong to the family G_s , they are of a similar nature and contain a distinguished central element ε of order 2 (distinguished arbitrarily in K). "Product" here means the central product obtained from the direct product by identifying the