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# HURWITZ-RADON MATRICES AND PERIODICITY MODULO 8

by Beno ECKMANN

## 0. INTRODUCTION

0.1. We consider complex $n \times n$ — matrices $A_1, A_2, \ldots, A_s$, either all unitary (case $U$) or all orthogonal (case $O$); they are called Hurwitz-Radon matrices, in short HR-matrices, if

(1) $$A_j^2 = -E, \quad A_jA_k + A_kA_j = 0, \quad j, k = 1, 2, \ldots, s, \; j \neq k;$$

$E$ or $E_n$ denotes the unit matrix. Such matrices are well-known to exist, even with entries $0, \pm 1, \pm i$ (case $U$) or $0, \pm 1$ (case $O$). The possible values of $n$ are multiples $mn_0, m = 1, 2, 3, \ldots$ where in case $U, n_0 = 2^{s/2}$ if $s$ is even, $n_0 = 2^{(s-1)/2}$ if $s$ is odd. In case $O, n_0 = 2^{(s-1)/2}$ if $s \equiv 7 \bmod 8$; $n_0 = 2^{s/2}$ if $s \equiv 0, 6$; $n_0 = 2^{(s+1)/2}$ if $s \equiv 1, 3, 5$; and $n_0 = 2^{(s+2)/2}$ if $s \equiv 2, 4 \bmod 8$.

If we put $A_0 = E$ the relations (1) are equivalent to

$$f_s(x_0, x_1, \ldots, x_s) = \sum_0^s x_jA_j$$

being a unitary, or orthogonal respectively, matrix for all real $x_j$ with $\sum_0^s x_j^2 = 1$. Thus $f_s$ can be considered as a map $S^s \to U$ via $U(n)$, or $S^s \to O$ via $O(n)$ where $U = \varinjlim U(k)$ and $O = \varinjlim O(k)$ are the infinite unitary and orthogonal groups. We also write $f_s$ for the homotopy class of that map, $f_s \in \pi_s(U)$ or $\pi_s(O)$. We recall that by the Bott periodicity theorems these groups are cyclic or 0.

THEOREM A. *If $A_1, A_2, \ldots, A_s$ are HR-matrices of minimal size $n = n_0(s)$ then $f_s$ is a generator of $\pi_s(U)$, or $\pi_s(O)$ respectively, $s = 0, 1, 2, \ldots$ .*

*Remark 1.* For $s = 0$ (empty set of HR-matrices) we have $f_0(x_0) = x_0(1)$, $x_0^2 = 1$; i.e., $f_0(1) = (1)$, $f_0(-1) = (-1)$, $f_0 \colon S^0 \to O(1) \to O$. For $s > 0$, $f_0 \colon S^s \to O$ clearly factors through $SO(n) \to SO$ ($U$ being connected, the analogue is irrelevant in the unitary case).

*Remark 2.* The problem originally solved by Hurwitz [H] and Radon [R] refers to the case $O$: One asks for complex bilinear forms $z = f(x, y) = (\sum_0^s x_j A_j) y$, where $z = (z_1, \ldots, z_n)$, $y = (y_1, \ldots, y_n)$, $x = (x_0, \ldots, x_s)$, such that

$$z_1^2 + \ldots + z_n^2 = (x_0^2 + \ldots + x_s^2)(y_1^2 + \ldots + y_n^2).$$

This means that $\sum_0^s x_j A_j$ is orthogonal, i.e. leaves invariant $\sum_0^n y_j^2$ except for the factor $\sum_0^s x_j^2$; and thus, since we may assume $A_0 = E$, that $A_1, \ldots, A_s$ is a set of orthogonal HR-matrices in the sense of (1).

The case $U$ refers to the analogous problem for the identity

$$|z_1|^2 + \ldots + |z_n|^2 = (x_0^2 + \ldots + x_s^2)(|y_1|^2 + \ldots + |y_n|^2)$$

where $y$ and $z$ are complex, and $x$ real.

0.2. The symplectic case: It is also of interest to consider HR-matrices, i.e. matrices fulfilling (1), which are symplectic. A linear combination $\sum_0^s x_j A_j$ of $2n \times 2n$-matrices with $A_0 = E$ is symplectic up to the factor $\sum_0^s x_j^2$ if and only if $A_1, \ldots, A_s$ is a set of symplectic HR-matrices (Proposition 4.1).

We restrict attention to unitary symplectic matrices, i.e., to the group $Sp(n) \subset U(2n)$, and write $Sp$ for the infinite symplectic group $\varinjlim Sp(k)$. With a set $A_1, \ldots, A_s$ of unitary symplectic HR-matrices, and $A_0 = E$, we associate the map $f_s(x_0, x_1, \ldots, x_s) = \sum_0^s x_j A_j$, $\sum_0^s x_j^2 = 1$, of $S^s$ into $Sp$ via $Sp(n)$; we also write $f_s$ for the corresponding element of $\pi_s(Sp)$, known to be 0 or cyclic.

THEOREM A′. *If* $A_1, \ldots, A_s$ *are unitary symplectic HR-matrices of minimal size* $2n_0$ *then* $f_s$ *is a generator of* $\pi_s(Sp)$.

0.3. The paper is organized as follows. We first recall (Section 1) that the HR-matrix problem can be formulated in terms of representations of certain finite group $G_s$, $s = 0, 1, 2, \ldots$ introduced by the author [E], and discuss these representations using the elegant description of [LS]. In Section 2 the "reduced" Grothendieck groups of representations $E_s^U$ and $E_s^O$ are computed; they turn out to be isomorphic to $\pi_s(U)$ and $\pi_s(O)$ respectively. Moreover a product is defined in the direct sum of the $E_s^U(E_s^O)$ turning it into a graded ring $E_*^U(E_*^O)$. The claim of Theorem A is proved in Section 3; we show that the maps $\phi: E_s^U \to \pi_s(U)$, $\psi: E_s^O \to \pi_s(O)$ given by the $f_s$ of 0.1 are isomorphisms. Using the product structure in $\pi_*(U)$ and $\pi_*(O)$ known from $K$-theory the proof reduces to simple verifications in low dimensions. The symplectic case is dealt with in Section 4. In Section 5 we make a remark concerning the "linearization phenomenon" for the homotopy groups of $U$, $O$ and $Sp$.

## 1. The groups $G_s$ and their representations

1.1. We will denote throughout by $G_s$ the group given by the presentation

$$G_s = <\varepsilon, a_1, ..., a_s \mid \varepsilon^2 = 1,\ a_j^2 = \varepsilon,\ a_j a_k = \varepsilon a_k a_j,\ j, k = 1, 2, ..., s, j \neq k> .$$

Clearly any set $A_1, ..., A_s$ of HR-matrices yields a (unitary or orthogonal) representation of $G_s$ of degree $n$ by $\varepsilon \mapsto -E$, $a_j \mapsto A_j$, $j = 1, 2, ..., s$. Conversely a representation of $G_s$ with $\varepsilon \mapsto -E$, in short an $\varepsilon$-representation, yields a set of $s$ HR-matrices. For the elementary properties of $G_s$ and its representations we refer to [E]. We just recall that the order of $G_s$ is $2^{s+1}$, that $\varepsilon$ is central, and that the irreducible unitary $\varepsilon$-representations of $G_s$ are of degree $2^{s/2}$ if $s$ is even (one equivalence class), of degree $2^{(s-1)/2}$ if $s$ is odd (two equivalence classes). These degrees are the minimal values $n_0$ in case $U$. As for the case $O$, one has to recall that a representation is equivalent to an orthogonal one if and only if it is equivalent to a real (and orthogonal) one. Thus, unless an irreducible unitary $\varepsilon$-representation is already real, one has to add its conjugate-complex representation, and the discussion of the various cases depending on $s$ yields the minimal values $n_0$ (case $O$) mentioned in the introduction; in other words, the degrees of the irreducible orthogonal $\varepsilon$-representations of $G_s$.

1.2. A very simple and useful scheme for studying the groups $G_s$ and their $\varepsilon$-representations (and many other things) has been deviced by T. Y. Lam and T. Smith [LS]. They have expressed the $G_s$ as products of very small and well-known groups. Namely $C = G_1$, the cyclic group of order 4; $Q = G_2$, the quaternionic group of order 8; $K$, the Klein 4-group; and $D$, the dihedral group of order 8. Although $K$ and $D$ do not belong to the family $G_s$, they are of a similar nature and contain a distinguished central element $\varepsilon$ of order 2 (distinguished arbitrarily in $K$). "Product" here means the central product obtained from the direct product by identifying the