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# HURWITZ-RADON MATRICES AND PERIODICITY MODULO 8

by Beno ECKMANN

## 0. INTRODUCTION

0.1. We consider complex  $n \times n$  - matrices  $A_1, A_2, \dots, A_s$ , either all unitary (case  $U$ ) or all orthogonal (case  $O$ ); they are called Hurwitz-Radon matrices, in short HR-matrices, if

$$(1) \quad A_j^2 = -E, \quad A_j A_k + A_k A_j = 0, \quad j, k = 1, 2, \dots, s, \quad j \neq k;$$

$E$  or  $E_n$  denotes the unit matrix. Such matrices are well-known to exist, even with entries  $0, \pm 1, \pm i$  (case  $U$ ) or  $0, \pm 1$  (case  $O$ ). The possible values of  $n$  are multiples  $mn_0, m = 1, 2, 3, \dots$  where in case  $U, n_0 = 2^{s/2}$  if  $s$  is even,  $n_0 = 2^{(s-1)/2}$  if  $s$  is odd. In case  $O, n_0 = 2^{(s-1)/2}$  if  $s \equiv 7 \pmod{8}$ ;  $n_0 = 2^{s/2}$  if  $s \equiv 0, 6$ ;  $n_0 = 2^{(s+1)/2}$  if  $s \equiv 1, 3, 5$ ; and  $n_0 = 2^{(s+2)/2}$  if  $s \equiv 2, 4 \pmod{8}$ .

If we put  $A_0 = E$  the relations (1) are equivalent to

$$f_s(x_0, x_1, \dots, x_s) = \sum_0^s x_j A_j$$

being a unitary, or orthogonal respectively, matrix for all real  $x_j$  with  $\sum_0^s x_j^2 = 1$ . Thus  $f_s$  can be considered as a map  $S^s \rightarrow U$  via  $U(n)$ , or  $S^s \rightarrow O$  via  $O(n)$  where  $U = \varinjlim U(k)$  and  $O = \varinjlim O(k)$  are the infinite unitary and orthogonal groups. We also write  $f_s$  for the homotopy class of that map,  $f_s \in \pi_s(U)$  or  $\pi_s(O)$ . We recall that by the Bott periodicity theorems these groups are cyclic or 0.

**THEOREM A.** *If  $A_1, A_2, \dots, A_s$  are HR-matrices of minimal size  $n = n_0(s)$  then  $f_s$  is a generator of  $\pi_s(U)$ , or  $\pi_s(O)$  respectively,  $s = 0, 1, 2, \dots$ .*

*Remark 1.* For  $s = 0$  (empty set of HR-matrices) we have  $f_0(x_0) = x_0(1)$ ,  $x_0^2 = 1$ ; i.e.,  $f_0(1) = (1)$ ,  $f_0(-1) = (-1)$ ,  $f_0: S^0 \rightarrow O(1) \rightarrow O$ . For  $s > 0$ ,  $f_0: S^s \rightarrow O$  clearly factors through  $SO(n) \rightarrow SO$  ( $U$  being connected, the analogue is irrelevant in the unitary case).

*Remark 2.* The problem originally solved by Hurwitz [H] and Radon [R] refers to the case  $O$ : One asks for complex bilinear forms  $z = f(x, y) = (\sum_0^s x_j A_j) y$ , where  $z = (z_1, \dots, z_n)$ ,  $y = (y_1, \dots, y_n)$ ,  $x = (x_0, \dots, x_s)$ , such that

$$z_1^2 + \dots + z_n^2 = (x_0^2 + \dots + x_s^2) (y_1^2 + \dots + y_n^2).$$

This means that  $\sum_0^s x_j A_j$  is orthogonal, i.e. leaves invariant  $\sum_0^n y_j^2$  except for the factor  $\sum_0^s x_j^2$ ; and thus, since we may assume  $A_0 = E$ , that  $A_1, \dots, A_s$  is a set of orthogonal HR-matrices in the sense of (1).

The case  $U$  refers to the analogous problem for the identity

$$|z_1|^2 + \dots + |z_n|^2 = (x_0^2 + \dots + x_s^2) (|y_1|^2 + \dots + |y_n|^2)$$

where  $y$  and  $z$  are complex, and  $x$  real.

0.2. The symplectic case: It is also of interest to consider HR-matrices, i.e. matrices fulfilling (1), which are symplectic. A linear combination  $\sum_0^s x_j A_j$  of  $2n \times 2n$ -matrices with  $A_0 = E$  is symplectic up to the factor  $\sum_0^s x_j^2$  if and only if  $A_1, \dots, A_s$  is a set of symplectic HR-matrices (Proposition 4.1).

We restrict attention to unitary symplectic matrices, i.e., to the group  $Sp(n) \subset U(2n)$ , and write  $Sp$  for the infinite symplectic group  $\varinjlim Sp(k)$ . With a set  $A_1, \dots, A_s$  of unitary symplectic HR-matrices, and  $A_0 = E$ , we associate the map  $f_s(x_0, x_1, \dots, x_s) = \sum_0^s x_j A_j$ ,  $\sum_0^s x_j^2 = 1$ , of  $S^s$  into  $Sp$  via  $Sp(n)$ ; we also write  $f_s$  for the corresponding element of  $\pi_s(Sp)$ , known to be 0 or cyclic.

**THEOREM A'.** *If  $A_1, \dots, A_s$  are unitary symplectic HR-matrices of minimal size  $2n_0$  then  $f_s$  is a generator of  $\pi_s(Sp)$ .*

0.3. The paper is organized as follows. We first recall (Section 1) that the HR-matrix problem can be formulated in terms of representations of certain finite group  $G_s$ ,  $s = 0, 1, 2, \dots$  introduced by the author [E], and discuss these representations using the elegant description of [LS]. In Section 2 the "reduced" Grothendieck groups of representations  $E_s^U$  and  $E_s^O$  are

computed; they turn out to be isomorphic to  $\pi_s(U)$  and  $\pi_s(O)$  respectively. Moreover a product is defined in the direct sum of the  $E_s^U(E_s^O)$  turning it into a graded ring  $E_*^U(E_*^O)$ . The claim of Theorem A is proved in Section 3; we show that the maps  $\phi: E_s^U \rightarrow \pi_s(U)$ ,  $\psi: E_s^O \rightarrow \pi_s(O)$  given by the  $f_s$  of 0.1 are isomorphisms. Using the product structure in  $\pi_*(U)$  and  $\pi_*(O)$  known from  $K$ -theory the proof reduces to simple verifications in low dimensions. The symplectic case is dealt with in Section 4. In Section 5 we make a remark concerning the "linearization phenomenon" for the homotopy groups of  $U$ ,  $O$  and  $Sp$ .

## 1. THE GROUPS $G_s$ AND THEIR REPRESENTATIONS

1.1. We will denote throughout by  $G_s$  the group given by the presentation

$$G_s = \langle \varepsilon, a_1, \dots, a_s \mid \varepsilon^2 = 1, a_j^2 = \varepsilon, a_j a_k = \varepsilon a_k a_j, j, k = 1, 2, \dots, s, j \neq k \rangle.$$

Clearly any set  $A_1, \dots, A_s$  of HR-matrices yields a (unitary or orthogonal) representation of  $G_s$  of degree  $n$  by  $\varepsilon \mapsto -E$ ,  $a_j \mapsto A_j$ ,  $j = 1, 2, \dots, s$ . Conversely a representation of  $G_s$  with  $\varepsilon \mapsto -E$ , in short an  $\varepsilon$ -representation, yields a set of  $s$  HR-matrices. For the elementary properties of  $G_s$  and its representations we refer to [E]. We just recall that the order of  $G_s$  is  $2^{s+1}$ , that  $\varepsilon$  is central, and that the irreducible unitary  $\varepsilon$ -representations of  $G_s$  are of degree  $2^{s/2}$  if  $s$  is even (one equivalence class), of degree  $2^{(s-1)/2}$  if  $s$  is odd (two equivalence classes). These degrees are the minimal values  $n_0$  in case  $U$ . As for the case  $O$ , one has to recall that a representation is equivalent to an orthogonal one if and only if it is equivalent to a real (and orthogonal) one. Thus, unless an irreducible unitary  $\varepsilon$ -representation is already real, one has to add its conjugate-complex representation, and the discussion of the various cases depending on  $s$  yields the minimal values  $n_0$  (case  $O$ ) mentioned in the introduction; in other words, the degrees of the irreducible orthogonal  $\varepsilon$ -representations of  $G_s$ .

1.2. A very simple and useful scheme for studying the groups  $G_s$  and their  $\varepsilon$ -representations (and many other things) has been devised by T. Y. Lam and T. Smith [LS]. They have expressed the  $G_s$  as products of very small and well-known groups. Namely  $C = G_1$ , the cyclic group of order 4;  $Q = G_2$ , the quaternionic group of order 8;  $K$ , the Klein 4-group; and  $D$ , the dihedral group of order 8. Although  $K$  and  $D$  do not belong to the family  $G_s$ , they are of a similar nature and contain a distinguished central element  $\varepsilon$  of order 2 (distinguished arbitrarily in  $K$ ). "Product" here means the central product obtained from the direct product by identifying the