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## HURWITZ-RADON MATRICES AND PERIODICITY MODULO 8

by Beno ECKMANN

### 0. INTRODUCTION

0.1. We consider complex  $n \times n$  – matrices  $A_1, A_2, \dots, A_s$ , either all unitary (case  $U$ ) or all orthogonal (case  $O$ ); they are called Hurwitz-Radon matrices, in short HR-matrices, if

$$(1) \quad A_j^2 = -E, \quad A_j A_k + A_k A_j = 0, \quad j, k = 1, 2, \dots, s, \quad j \neq k;$$

$E$  or  $E_n$  denotes the unit matrix. Such matrices are well-known to exist, even with entries  $0, \pm 1, \pm i$  (case  $U$ ) or  $0, \pm 1$  (case  $O$ ). The possible values of  $n$  are multiples  $mn_0$ ,  $m = 1, 2, 3, \dots$  where in case  $U$ ,  $n_0 = 2^{s/2}$  if  $s$  is even,  $n_0 = 2^{(s-1)/2}$  if  $s$  is odd. In case  $O$ ,  $n_0 = 2^{(s-1)/2}$  if  $s \equiv 7 \pmod{8}$ ;  $n_0 = 2^{s/2}$  if  $s \equiv 0, 6$ ;  $n_0 = 2^{(s+1)/2}$  if  $s \equiv 1, 3, 5$ ; and  $n_0 = 2^{(s+2)/2}$  if  $s \equiv 2, 4 \pmod{8}$ .

If we put  $A_0 = E$  the relations (1) are equivalent to

$$f_s(x_0, x_1, \dots, x_s) = \sum_0^s x_j A_j$$

being a unitary, or orthogonal respectively, matrix for all real  $x_j$  with  $\sum_0^s x_j^2 = 1$ . Thus  $f_s$  can be considered as a map  $S^s \rightarrow U$  via  $U(n)$ , or  $S^s \rightarrow O$  via  $O(n)$  where  $U = \lim_{\rightarrow} U(k)$  and  $O = \lim_{\rightarrow} O(k)$  are the infinite unitary and orthogonal groups. We also write  $f_s$  for the homotopy class of that map,  $f_s \in \pi_s(U)$  or  $\pi_s(O)$ . We recall that by the Bott periodicity theorems these groups are cyclic or 0.

**THEOREM A.** *If  $A_1, A_2, \dots, A_s$  are HR-matrices of minimal size  $n = n_0(s)$  then  $f_s$  is a generator of  $\pi_s(U)$ , or  $\pi_s(O)$  respectively,  $s = 0, 1, 2, \dots$ .*

**Remark 1.** For  $s = 0$  (empty set of HR-matrices) we have  $f_0(x_0) = x_0(1)$ ,  $x_0^2 = 1$ ; i.e.,  $f_0(1) = (1)$ ,  $f_0(-1) = (-1)$ ,  $f_0: S^0 \rightarrow O(1) \rightarrow O$ . For  $s > 0$ ,  $f_s: S^s \rightarrow O$  clearly factors through  $SO(n) \rightarrow SO$  ( $U$  being connected, the analogue is irrelevant in the unitary case).

*Remark 2.* The problem originally solved by Hurwitz [H] and Radon [R] refers to the case  $O$ : One asks for complex bilinear forms  $z = f(x, y) = (\sum_0^s x_j A_j) y$ , where  $z = (z_1, \dots, z_n)$ ,  $y = (y_1, \dots, y_n)$ ,  $x = (x_0, \dots, x_s)$ , such that

$$z_1^2 + \dots + z_n^2 = (x_0^2 + \dots + x_s^2)(y_1^2 + \dots + y_n^2).$$

This means that  $\sum_0^s x_j A_j$  is orthogonal, i.e. leaves invariant  $\sum_0^n y_j^2$  except for the factor  $\sum_0^s x_j^2$ ; and thus, since we may assume  $A_0 = E$ , that  $A_1, \dots, A_s$  is a set of orthogonal HR-matrices in the sense of (1).

The case  $U$  refers to the analogous problem for the identity

$$|z_1|^2 + \dots + |z_n|^2 = (x_0^2 + \dots + x_s^2)(|y_1|^2 + \dots + |y_n|^2)$$

where  $y$  and  $z$  are complex, and  $x$  real.

0.2. The symplectic case: It is also of interest to consider HR-matrices, i.e. matrices fulfilling (1), which are symplectic. A linear combination  $\sum_0^s x_j A_j$  of  $2n \times 2n$ -matrices with  $A_0 = E$  is symplectic up to the factor  $\sum_0^s x_j^2$  if and only if  $A_1, \dots, A_s$  is a set of symplectic HR-matrices (Proposition 4.1).

We restrict attention to unitary symplectic matrices, i.e., to the group  $Sp(n) \subset U(2n)$ , and write  $Sp$  for the infinite symplectic group  $\lim_{\rightarrow} Sp(k)$ . With a set  $A_1, \dots, A_s$  of unitary symplectic HR-matrices, and  $A_0 = E$ , we associate the map  $f_s(x_0, x_1, \dots, x_s) = \sum_0^s x_j A_j$ ,  $\sum_0^s x_j^2 = 1$ , of  $S^s$  into  $Sp$  via  $Sp(n)$ ; we also write  $f_s$  for the corresponding element of  $\pi_s(Sp)$ , known to be 0 or cyclic.

**THEOREM A'.** *If  $A_1, \dots, A_s$  are unitary symplectic HR-matrices of minimal size  $2n_0$  then  $f_s$  is a generator of  $\pi_s(Sp)$ .*

0.3. The paper is organized as follows. We first recall (Section 1) that the HR-matrix problem can be formulated in terms of representations of certain finite group  $G_s$ ,  $s = 0, 1, 2, \dots$  introduced by the author [E], and discuss these representations using the elegant description of [LS]. In Section 2 the “reduced” Grothendieck groups of representations  $E_s^U$  and  $E_s^O$  are

computed; they turn out to be isomorphic to  $\pi_s(U)$  and  $\pi_s(O)$  respectively. Moreover a product is defined in the direct sum of the  $E_s^U(E_s^O)$  turning it into a graded ring  $E_*^U(E_*^O)$ . The claim of Theorem A is proved in Section 3; we show that the maps  $\phi: E_s^U \rightarrow \pi_s(U)$ ,  $\psi: E_s^O \rightarrow \pi_s(O)$  given by the  $f_s$  of 0.1 are isomorphisms. Using the product structure in  $\pi_*(U)$  and  $\pi_*(O)$  known from  $K$ -theory the proof reduces to simple verifications in low dimensions. The symplectic case is dealt with in Section 4. In Section 5 we make a remark concerning the “linearization phenomenon” for the homotopy groups of  $U$ ,  $O$  and  $Sp$ .

### 1. THE GROUPS $G_s$ AND THEIR REPRESENTATIONS

1.1. We will denote throughout by  $G_s$  the group given by the presentation

$$G_s = \langle \varepsilon, a_1, \dots, a_s \mid \varepsilon^2 = 1, a_j^2 = \varepsilon, a_j a_k = \varepsilon a_k a_j, j, k = 1, 2, \dots, s, j \neq k \rangle.$$

Clearly any set  $A_1, \dots, A_s$  of HR-matrices yields a (unitary or orthogonal) representation of  $G_s$  of degree  $n$  by  $\varepsilon \mapsto -E$ ,  $a_j \mapsto A_j$ ,  $j = 1, 2, \dots, s$ . Conversely a representation of  $G_s$  with  $\varepsilon \mapsto -E$ , in short an  $\varepsilon$ -representation, yields a set of  $s$  HR-matrices. For the elementary properties of  $G_s$  and its representations we refer to [E]. We just recall that the order of  $G_s$  is  $2^{s+1}$ , that  $\varepsilon$  is central, and that the irreducible unitary  $\varepsilon$ -representations of  $G_s$  are of degree  $2^{s/2}$  if  $s$  is even (one equivalence class), of degree  $2^{(s-1)/2}$  if  $s$  is odd (two equivalence classes). These degrees are the minimal values  $n_0$  in case  $U$ . As for the case  $O$ , one has to recall that a representation is equivalent to an orthogonal one if and only if it is equivalent to a real (and orthogonal) one. Thus, unless an irreducible unitary  $\varepsilon$ -representation is already real, one has to add its conjugate-complex representation, and the discussion of the various cases depending on  $s$  yields the minimal values  $n_0$  (case  $O$ ) mentioned in the introduction; in other words, the degrees of the irreducible orthogonal  $\varepsilon$ -representations of  $G_s$ .

1.2. A very simple and useful scheme for studying the groups  $G_s$  and their  $\varepsilon$ -representations (and many other things) has been devised by T. Y. Lam and T. Smith [LS]. They have expressed the  $G_s$  as products of very small and well-known groups. Namely  $C = G_1$ , the cyclic group of order 4;  $Q = G_2$ , the quaternionic group of order 8;  $K$ , the Klein 4-group; and  $D$ , the dihedral group of order 8. Although  $K$  and  $D$  do not belong to the family  $G_s$ , they are of a similar nature and contain a distinguished central element  $\varepsilon$  of order 2 (distinguished arbitrarily in  $K$ ). “Product” here means the central product obtained from the direct product by identifying the

two  $\varepsilon$ 's. The expression for the  $G_s$  then is as follows, displaying a fundamental periodicity modulo 8:

(2)	$s$	0	1	2	3	4	5	6	7	8	9	...
	$G_s$	$\mathbf{Z}/2$	$C$	$Q$	$QK$	$QD$	$D^2C$	$D^3$	$D^3K$	$D^4$	$D^4C$	...

and  $G_{s+8} = D^4G_s$ .

The tensor product of  $\varepsilon$ -representations of two of the groups  $G_s, K, D$  is an  $\varepsilon$ -representation of their product above, and all  $\varepsilon$ -representations of the  $G_s$  can be obtained in that explicit way from those of  $C, Q, K, D$ , which are well-known. This yields, in particular, the characters  $\chi$  and the Schur indices  $I$  of the irreducible unitary  $\varepsilon$ -representation (the Schur index  $I = 1$  if the representation is equivalent to a real one; if it is not,  $I = -1$  if it is equivalent to the conjugate-complex one,  $I = 0$  otherwise). Both  $\chi$  and  $I$  behave multiplicatively with respect to the central product.

1.3. The Schur indices of the irreducible  $\varepsilon$ -representations are: 0 for  $C = G_1$ ,  $-1$  for  $Q = G_2$ , and 1 for  $K$  and  $D$  (two equivalence classes for  $K$ , one for  $D$ ). This yields the Schur indices  $I_s$  of the irreducible  $\varepsilon$ -representations of the  $G_s$ , as listed in (2) below; we further list the numbers  $v_s^U$  of inequivalent unitary, and  $v_s^O$  of inequivalent orthogonal irreducible  $\varepsilon$ -representations, and the respective degrees  $d_s^U, d_s^O$ . Note that  $I_s$  is periodic with period 8, and  $d_s^O$  is periodic with period 8 in the sense that  $d_{s+8}^O = 16d_s^O$ . Finally we include in the same table the Grothendieck groups  $D_s^U$  and  $D_s^O$  of (equivalence classes of) irreducible  $\varepsilon$ -representations of  $G_s$ , with respect to the direct sum of representations.

(3)	$s$	0	1	2	3	4	5	6	7	8	9	...
	$I_s$	1	0	-1	-1	-1	0	1	1	1	0	...
	$v_s^U$	1	2	1	2	1	2	1	2	1	2	
	$v_s^O$	1	1	1	2	1	1	1	2	1	1	
	$d_s^U$	1	1	2	2	4	4	8	8	16	16	
	$d_s^O$	1	2	4	4	8	8	8	8	16	32	
	$D_s^U$	$\mathbf{Z}$	$\mathbf{Z} \oplus \mathbf{Z}$									
	$D_s^O$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	

The values of  $d_s^O$  follow immediately from the  $I_s$  and the  $d_s^U$ . The values  $n_0$  for the case  $O$ , as given in the Introduction, are the  $d_s^O$ .

## 2. THE REDUCED $\varepsilon$ -REPRESENTATION RING

2.1. For all  $s \geq 0$  the group  $G_s$  is the subgroup of  $G_{s+1}$  obtained by omitting the generator  $a_{s+1}$ ; let  $h_s: G_s \rightarrow G_{s+1}$  be the embedding homomorphism. Via  $h_s$  we can restrict an  $\varepsilon$ -representation of  $G_{s+1}$  to  $G_s$ , which in terms of HR-matrices means omitting  $A_{s+1}$ .

Let  $h_s^*: D_{s+1}^U \rightarrow D_s^U$  be the corresponding homomorphism of Grothendieck groups, and  $E_s^U = D_s^U/h_s^* D_{s+1}^U$  the “reduced” groups; similarly  $E_s^O = D_s^O/h_s^* D_{s+1}^O$ . They can easily be computed by means of the characters of  $\varepsilon$ -representations, as follows.

For  $Q$  and  $D$  the character of an irreducible unitary  $\varepsilon$ -representation is 0 except on 1 and  $\varepsilon$ . For  $C$  and  $K$  it is  $\neq 0$  on all 4 elements; on the essential generator ( $\neq \varepsilon$ ) of  $C$  it is  $+i$  or  $-i$  for the two inequivalent representations, and  $+1$  or  $-1$  in the case of  $K$ . For  $G_s$ ,  $s$  even, we infer from the table (2) that the character is 0 except on 1,  $\varepsilon$ . For  $G_s$ ,  $s$  odd, the character is 0 except on 1,  $\varepsilon$  and two further elements  $z, \varepsilon z$ ; on these the two inequivalent  $\varepsilon$ -representations differ just by the sign of the character.

If  $s$  is even,  $d_{s+1}^U = d_s^U = 2^{s/2}$ ; thus the restriction of an irreducible  $\varepsilon$ -representation must be irreducible, whence  $h_s^* D_{s+1}^U = D_s^U$ ,  $E_s^U = 0$ . If  $s$  is odd,  $d_{s+1}^U = 2d_s^U = 2^{(s+1)/2}$ ; thus the restriction is the sum of two irreducible  $\varepsilon$ -representations, and since the character is 0 (except on 1,  $\varepsilon$ ) these two must be inequivalent. Therefore  $h_s^* D_{s+1}^U$  is the “diagonal” of  $D_s^U = \mathbf{Z} \oplus \mathbf{Z}$ , and  $E_s^U = \mathbf{Z}$ ; its generator  $\rho_s$  is represented by either of the two inequivalent irreducible  $\varepsilon$ -representations of  $G_s$ ,  $-\rho_s$  by the other one.

In the orthogonal case the  $E_s^O$  are computed similarly from (3). Since  $d_1^O = 2$  and  $d_0^O = 1$ , the restriction from  $D_1^O$  to  $D_0^O$  yields twice the generator, and  $E_0^O = \mathbf{Z}/2$ ; the same argument holds for  $s \equiv 0 \pmod{8}$ ,  $d_{s+1}^O = 2d_s^O$ . Since  $d_2^O = 4$  and  $d_1^O = 2$ , we get  $E_1^O = \mathbf{Z}/2$ . From  $d_3^O = d_2^O = 4$  we get  $E_2^O = 0$ . As for  $s = 3$ , the character argument shows that  $h_3^* D_4^O$  = diagonal of  $D_3^O (= \mathbf{Z} \oplus \mathbf{Z})$ , and  $E_3^O = \mathbf{Z}$ . For  $s = 4, 5, 6$  the dimensions  $d_{s+1}^O = d_s^O$  show that  $E_4^O = E_5^O = E_6^O = 0$ . For  $s = 7$ , the character argument yields  $h_7^* D_8^O$  = diagonal of  $D_7^O (= \mathbf{Z} \oplus \mathbf{Z})$ , and  $E_7^O = \mathbf{Z}$ . Finally one has, for all  $s$ ,  $E_{s+8}^O \cong E_s^O$ .

These results are summarized in the table

(4)	$s$	0	1	2	3	4	5	6	7	8	9	...
$E_s^U$		0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	
$E_s^O$		$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	

According to the Bott periodicity theorems the above table is just that of the  $\pi_s(U)$  and  $\pi_s(O)$ ,  $s = 0, 1, 2, \dots$ . Before studying the relation as stated in Theorem A we establish product structures in the reduced Grothendieck groups of  $\varepsilon$ -representations, i.e., of HR-matrices.

2.2. We consider HR-matrices  $A_1, A_2, \dots, A_s \in U(n)$  and put, for

$$x = (x_0, x_1, \dots, x_s) \in \mathbf{R}^{s+1}$$

and  $A_0 = E_n$  ( $n \times n$  unit matrix)

$$f(x) = \sum_0^s x_j A_j.$$

For all  $x$  with  $|x| = 1$ ,  $f(x)$  is a unitary matrix: this is, as mentioned in the Introduction, precisely the meaning of the HR-matrix relations (1).

Let further  $B_1, B_2, \dots, B_t \in U(m)$  be HR-matrices, and for

$$y = (y_0, y_1, \dots, y_t) \in \mathbf{R}^{t+1}, \quad B_0 = E_m,$$

$$g(y) = \sum_0^t y_k B_k;$$

$g(y) \in U(m)$  for all  $y$  with  $|y| = 1$ . We define  $F$  by

$$F(x, y) = \begin{pmatrix} f(x) \otimes E_m & E_n \otimes g(y) \\ -E_n \otimes \overline{g(y)}^T & \overline{f(x)}^T \otimes E_m \end{pmatrix}.$$

One immediately checks that  $F(x, y) \bar{F}^T(x, y) = (|x|^2 + |y|^2) E_{2nm}$ . Thus  $F(x, y) \in U(2nm)$  for all  $(x, y) \in \mathbf{R}^{s+t+2}$  with  $|x|^2 + |y|^2 = 1$ . Since the coefficient matrix of  $x_0$  is  $E_{2nm}$  the coefficient matrices of  $x_1, \dots, x_s, y_0, \dots, y_t$  constitute a set of  $s + t + 1$  HR-matrices  $\in U(2nm)$ . They are, explicitly,

$$(5) \quad \begin{pmatrix} A_j \otimes E_m & 0 \\ 0 & -A_j \otimes E_m \end{pmatrix}, \begin{pmatrix} 0 & E_{nm} \\ -E_{nm} & 0 \end{pmatrix}, \begin{pmatrix} 0 & E_n \otimes B_k \\ E_n \otimes B_k & 0 \end{pmatrix}$$

with  $j = 1, \dots, s$  and  $k = 1, \dots, t$ . In other words, we have a product of  $\varepsilon$ -representations of  $G_s$  and  $G_t$

$$D_s^U \times D_t^U \xrightarrow{\sim} D_{s+t+1}^U.$$

Since addition in  $D_s^U$  is by the direct sum of  $\varepsilon$ -representations this product is clearly distributive. Associativity (up to equivalence) is easily checked. We thus get a ring structure in  $D_*^U = \bigoplus_{s=-1}^{\infty} D_s^U$ ; we have added the term  $D_{-1}^U = \mathbf{Z}$  generated by the ring unit. The ring  $D_*^U$  is graded if the grading is by  $s + 1$  for  $D_s$ .

From the HR-matrices (5) of the product one notes that if one of the two factors is restricted from  $D_*^U$  so is the product; i.e.,  $h*D_*^U$  is a (graded) ideal in  $D_*^U$ , and we get a (graded) ring structure in  $D_*^U/h*D_*^U = E_*^U$ .

The same procedure yields, of course, a (graded) ring structure in  $E_*^O = \bigoplus_{s=-1}^{\infty} E_s^O$ , with grading  $s + 1$  for  $E_s^O$ . In 2.3 and 2.4 below these rings are described explicitly.

*Remark 2.1.* An easy computation shows that the rings  $E_*^U$  and  $E_*^O$  are anticommutative with respect to the grading, i.e., commutative except for the factor  $(-1)^{(s+1)(t+1)}$ . This will not really be used since the  $E_s^U$  and  $E_s^O$  are all 0,  $\mathbf{Z}$  or  $\mathbf{Z}/2$ . We just note that in the case  $\mathbf{Z}$ , with generator  $\rho_s$ ,  $-\rho_s$  is given by the other equivalence class of irreducible  $\varepsilon$ -representations, see 2.1.

### 2.3. The ring $E_*^U$ .

The generator  $\rho_s$  of  $E_s^U$ , given by an irreducible unitary  $\varepsilon$ -representation of  $G_s$ , has degree  $2^{s/2}$  if  $s$  is even,  $2^{(s-1)/2}$  if  $s$  is odd. The product  $\rho_s \rho_t \in E_{s+t+1}^U$  has degree

$$\begin{aligned} 2^{(s+t+2)/2} &\quad \text{if } s \text{ and } t \text{ are even,} \\ 2^{(s+t+1)/2} &\quad \text{if } s \text{ is even, } t \text{ odd, or vice-versa,} \\ 2^{(s+t)/2} &\quad \text{if } s \text{ and } t \text{ are odd.} \end{aligned}$$

Thus, unless both  $s$  and  $t$  are even, the product is irreducible, i.e.,  $\rho_s \rho_t = \pm \rho_{s+t+1}$ . After choice of  $\rho_1 \in E_1^U$  we can choose  $\rho_3 = \rho_1^2$ ,  $\rho_5 = \rho_1 \rho_3 = \rho_3 \rho_1 = \rho_1^3$ , ..., and for all odd  $s = 2r - 1$ ,  $\rho_s = \rho_1^r$ ; for even  $s$ ,  $E_s^U = 0$ .

**PROPOSITION 2.2.** *The product with  $\rho_1 \in E_1^U$  is an isomorphism  $E_s^U \cong E_{s+2}^U$  for all  $s$ . For odd  $s = 2l - 1$  we choose*

$$\rho_{2l-1} = \rho_1^l, l = 1, 2, 3, \dots .$$

**THEOREM 2.3.**  $E_*^U$  is the polynomial ring  $\mathbf{Z}[\rho_1]$ .

#### 2.4. THE RING $E_*^O$ .

We denote by  $\sigma_s$  the generator of  $E_s^O$  ( $= 0$  if  $s \equiv 2, 4, 5, 6$  modulo 8; determined up to sign if  $s \equiv 3, 7$  modulo 8 where  $E_s^O = \mathbf{Z}$ ).

The generator  $\rho_7$  ( $= \rho_1^4 \in E_7^U$ ) can be given by a real  $\varepsilon$ -representation of degree 8 which we can use as generator  $\sigma_7 \in E_7^O$ . The ring homomorphism  $\Phi: E_*^O \rightarrow E_*^U$  induced by the embedding  $O \rightarrow U$ ,  $\Phi(\sigma_7) = \rho_7$ , is thus an isomorphism  $E_7^O \cong E_7^U$ . In  $E_*^O$  the degree of  $\sigma_7 \sigma_s \in E_{s+8}^O$  is  $16d_s^O = d_{s+8}^O$ . Hence  $\sigma_7 \sigma_s$  is irreducible, i.e.,  $= \pm \sigma_{s+8}$  for all  $s$ . In particular we can choose  $\sigma_{15} = \sigma_7^2$ ,  $\sigma_{23} = \sigma_7^3$ , ...,  $\sigma_{8r-1} = \sigma_7^r$ .

**PROPOSITION 2.4.** The isomorphism  $E_s^O \cong E_{s+8}^O$  can be given by the product with  $\sigma_7 \in E_7^O$ .

**PROPOSITION 2.5.**  $\sigma_7 \in E_7^O$  generates a subring of  $E_*^O$  which is the polynomial ring  $\mathbf{Z}[\sigma_7]$ .

We further note that  $\sigma_3 \in E_3^O$  is mapped by  $\Phi$  to  $2\rho_3 \in E_3^U$ . From  $\Phi(\sigma_3^2) = 4\rho_3^2 = 4\rho_7 = \Phi(4\sigma_7)$  we infer that  $\sigma_3^2 = 4\sigma_7$ . As for  $\sigma_0 \in E_0^O$ , it is of degree 1 and order 2, and  $\sigma_0^2 \in E_1^O$  is of degree 2 and order 2, i.e.,  $\sigma_0^2 = \sigma_1$ . Of course  $\sigma_0^3 = 0$ .

In summary:

**THEOREM 2.6.**  $E_*^O$  is the commutative ring, graded by  $s+1$  for  $E_s^O$ , generated by  $\sigma_0, \sigma_3, \sigma_7$  with the only relations  $2\sigma_0 = 0$ ,  $\sigma_0^3 = 0$ ,  $\sigma_3^2 = 4\sigma_7$ .

### 3. THE HOMOTOPY GROUPS OF $U$ AND $O$

**3.1.** We will deal explicitly with the unitary case. The orthogonal case can be treated in almost exactly the same way; any additional arguments will be mentioned wherever necessary.

In the Introduction 0.1 we associated with a set of  $s$  unitary  $n \times n$  HR-matrices, i.e., with an  $\varepsilon$ -representation of  $G_s$ , a map  $f: S^s \rightarrow U$  of the  $s$ -sphere  $S^s \subset \mathbf{R}^{s+1}$  into the infinite unitary group  $U$  via  $U(n)$ . Since conjugation is homotopic to the identity, equivalent representations yield homotopic maps  $f$  (in the orthogonal case, we have to observe that conjugation can be made with a matrix from the identity component). The map  $\phi: D_s^U \rightarrow \pi_s(U)$  thus obtained is a homomorphism; indeed, homotopy group addition of  $f$  and  $f'$  in  $\pi_s(U(n))$  can be replaced by multiplication in

$U(n)$ ; this is homotopic in  $U(2n)$  to the map  $\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}$ , and on the other hand addition in  $D_s^U$  is defined through the direct sum of representations.

If the  $\varepsilon$ -representation is restricted from  $D_{s+1}^U$ , i.e., if the set of HR-matrices belongs to a set of  $s+1$  HR-matrices,  $f$  extends to a map  $S^{s+1} \rightarrow U$  and is thus nullhomotopic. The homomorphism  $\phi$  therefore induces a homomorphism  $E_s^U \rightarrow \pi_s(U)$ , again written  $\phi$ . The analogue  $E_s^O \rightarrow \pi_s(O)$  will be denoted by  $\psi$ . The groups  $E_s^U$  and  $E_s^O$  are 0 or cyclic generated by irreducible  $\varepsilon$ -representations, i.e., by HR-matrices of minimal size. Our claim, Theorem A, can therefore be reformulated as follows.

**THEOREM B.** *The homomorphisms  $\phi: E_s^U \rightarrow \pi_s(U)$  and  $\psi: E_s^O \rightarrow \pi_s(O)$  are isomorphisms,  $s = 0, 1, 2, \dots$ .*

3.2. For small values of  $s$  the claim is easily checked.

*Case U*

$s = 1$ :  $E_1^U$  can be generated by one HR-matrix  $A_1 = (i)$ . Thus

$$f(x_0, x_1) = (x_0 + ix_1) \in U(1)$$

if  $x_0^2 + x_1^2 = 1$ . This is a generator of  $\pi_1(U(1)) \cong \pi_1(U) = \mathbf{Z}$ .

$s = 3$ :  $E_3^U$  is generated by 3 HR-matrices

$$A_1 = \begin{pmatrix} i & \\ & -i \end{pmatrix}, \quad A_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad A_3 = \begin{pmatrix} & i \\ i & \end{pmatrix}.$$

Thus

$$f(x_0, x_1, x_2, x_3) = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \in SU(2)$$

if  $\sum_0^3 x_j^2 = 1$ . This is a generator of  $\pi_3(SU(2)) [= \pi_3(S^3)] \cong \pi_3(U) = \mathbf{Z}$ .

*Case O*

$s = 0$ : Empty set of HR-matrices,  $f(x_0) = (x_0) \in O(1)$  if  $x_0^2 = 1$ ,  $x_0 = \pm 1$ . This is a generator of  $\pi_0(O(1)) \cong \pi_0(O) = \mathbf{Z}/2$ .

$s = 1$ :  $E_1^O$  is generated by one HR-matrix  $A_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ . Thus

$$f(x_0, x_1) = \begin{pmatrix} x_0 & x_1 \\ -x_1 & x_0 \end{pmatrix} \in SO(2)$$

if  $x_0^2 + x_1^2 = 1$ . This is a generator of  $\pi_1(SO(2)) = \mathbf{Z}$ ; as a map  $S^1 \rightarrow SO(3)$  it is a generator of  $\pi_1(SO(3)) \cong \pi_1(O) = \mathbf{Z}/2$ .

$s = 3$ :  $E_3^O$  is generated by three  $4 \times 4$  HR-matrices which yield

$$f(x_0, x_1, x_2, x_3) = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_1 & x_0 & x_3 & -x_2 \\ -x_2 & -x_3 & x_0 & x_1 \\ -x_3 & x_2 & -x_1 & x_0 \end{pmatrix} \in SO(4)$$

if  $\sum_0^3 x_j^2 = 1$ . This is a map  $S^3 \rightarrow SO(4)$  which is well-known to become, under  $SO(4) \rightarrow SO(5)$ , a generator of  $\pi_3(SO(5)) \cong \pi_3(O) = \mathbf{Z}$ .

3.3. The proof of Theorem B becomes very simple if  $\phi$  and  $\psi$  are turned into ring homomorphisms  $E_*^U \rightarrow \pi_*(U) = \bigoplus_{-1}^{\infty} \pi_s(U)$  ( $\pi_{-1} = \mathbf{Z}$  generated by the ring unit) and  $E_*^O \rightarrow \pi_*(O)$ . For this purpose we have to define a product in  $\pi_*(U)$  and  $\pi_*(O)$ , graded by  $s + 1$  for  $\pi_s$ . This is done by extending the product introduced in 2.2 from linear maps  $f: S^s \rightarrow U$  or  $O$  to arbitrary continuous maps.

Given a continuous map  $f: S^s \rightarrow U$  via  $U(n)$ ,

$$S^s = \{x = (x_0, x_1, \dots, x_s) \in \mathbf{R}^{s+1} \text{ with } |x| = 1\},$$

we extend it to  $f_0: \mathbf{R}^{s+1} \rightarrow M_n(\mathbf{C})$  by  $f_0(x) = |x| f\left(\frac{x}{|x|}\right)$ ,  $f_0(0) = 0$ .

Similarly for  $g: S^t \rightarrow U$  via  $U(m)$ ,  $S^t = \{y \in \mathbf{R}^{t+1} \text{ with } |y| = 1\}$ . Then

$$F(x, y) = \begin{pmatrix} f_0(x) \otimes E_m & E_n \otimes g_0(y) \\ -E_n \otimes \overline{g_0(y)}^T & \overline{f_0(x)}^T \otimes E_m \end{pmatrix}$$

is a unitary  $2nm \times 2nm$  matrix for all  $(x, y) \in \mathbf{R}^{s+t+2}$  with  $|x|^2 + |y|^2 = 1$  and thus defines a map  $F: S^{s+t+1} \rightarrow U$  via  $U(2nm)$ . Homotopic maps  $f$ , or  $g$  respectively, yield homotopic  $F$  and we obtain a product  $F = f \cup g$

$$\pi_s(U) \times \pi_t(U) \xrightarrow{\cup} \pi_{s+t+1}(U).$$

From the description of homotopy group addition in  $\pi_s(U)$  as given above in 3.1 one easily checks that  $f \cup g$  is distributive. Thus  $\pi_*(U)$  is a ring, and so is  $\pi_*(O)$ , graded by  $s + 1$  for  $\pi_s(U)$  or  $\pi_s(O)$ .

3.4. Bott periodicity is usually expressed in terms of complex and real  $K$ -theory. We thus use the isomorphisms

$$\pi_s(U) \cong \tilde{K}_{\mathbf{C}}(S^{s+1}) \quad \text{and} \quad \pi_s(O) \cong \tilde{K}_{\mathbf{R}}(S^{s+1}).$$

We recall that  $\pi_s(U) \cong \tilde{K}_{\mathbf{C}}(S^{s+1})$  is obtained through  $\pi_s(U) \cong K_{\mathbf{C}}(B^{s+1}, S^s)$  where  $B^{s+1}$  is the unit ball  $\{x \in \mathbf{R}^{s+1}, |x| \leq 1\}$ ; the element corresponding to  $f \in \pi_s(U)$  is given by two (trivial)  $\mathbf{C}$ -vector bundles over  $B^{s+1}$ , identified on  $S^s$  by means of  $f$ . It will not come as a surprise that  $f \cup g$  above corresponds to the  $\cup$ -product

$$K_{\mathbf{C}}(B^{s+1}, S^s) \times K_{\mathbf{C}}(B^{t+1}, S^t) \rightarrow K_{\mathbf{C}}(B^{s+t+2}, S^{s+t+1})$$

given by the external tensor product of bundles. Indeed the map  $f \cup g = F: S^{s+t+1} \rightarrow U$  via  $U(2nm)$  can be interpreted as follows: One decomposes  $S^{s+t+1} \subset \mathbf{R}^{s+t+2}$  (coordinates  $x_0, x_1, \dots, x_s, y_0, y_1, \dots, y_t$  with  $|x|^2 + |y|^2 = 1$ ) into  $\{|x|^2 \leq \frac{1}{2}, |y|^2 \geq \frac{1}{2}\}$  homeomorphic to  $B^{s+1} \times S^t$  and  $\{|x|^2 \geq \frac{1}{2}, |y|^2 \leq \frac{1}{2}\}$  homeomorphic to  $S^s \times B^{t+1}$ ; the map  $F$  is

$$\begin{aligned} \begin{pmatrix} f(x) \otimes E_m & 0 \\ 0 & \overline{f(x)}^T \otimes E_m \end{pmatrix} & \quad \text{on} \quad S^s \times (0), \text{ i.e. } y = 0, |x| = 1, \\ \begin{pmatrix} 0 & E_n \otimes g(y) \\ -E_n \otimes \overline{g(y)}^T & 0 \end{pmatrix} & \quad \text{on} \quad (0) \times S^t, \text{ i.e. } x = 0, |y| = 1. \end{aligned}$$

Under  $K_{\mathbf{C}}(B^{s+1}, S^s) \cong \tilde{K}_{\mathbf{C}}(S^{s+1})$  one then has a graded ring structure in  $\bigoplus_{-1}^{\infty} \tilde{K}_{\mathbf{C}}(S^{s+1})$  isomorphic to  $\pi_*(U)$ . According to the Bott periodicity theorem (see [K], p. 123) this ring is the polynomial ring  $\mathbf{Z}[a]$  generated by the generator of  $\tilde{K}_{\mathbf{C}}(S^2)$ ; i.e.,  $\pi_*(U)$  is the polynomial ring generated by the generator  $a$  of  $\pi_1(U)$ .

Similarly,  $\pi_*(O)$  is the commutative ring with generators  $b_0 \in \pi_0(O)$ ,  $b_3 \in \pi_3(O)$ ,  $b_7 \in \pi_7(O)$  with relations  $2b_0 = 0$ ,  $b_0^3 = 0$ ,  $b_3^2 = 4b_7$  ([K], p. 156-157).

To prove Theorem B we therefore only have to show:

Case  $U$ .  $\rho_1 \in E_1^U$  is mapped by  $\phi$  to  $a \in \pi_1(U)$ .

Case  $O$ .  $\sigma_0 \in E_0^O$  is mapped by  $\psi$  to  $b_0 \in \pi_0(O)$  and  $\sigma_3 \in E_3^O$  to  $b_3 \in \pi_3(O)$

This has already been done in 3.2.

## 4. SYMPLECTIC HR-MATRICES

4.1. Symplectic matrices  $A$  leave invariant the bilinear form with coefficient matrix  $J = \begin{pmatrix} E_n \\ -E_n \end{pmatrix}$ ; i.e.,  $A^TJA = J$ . With respect to the HR-matrix relations (1) they behave exactly like orthogonal or unitary matrices:

PROPOSITION 4.1. *Let  $A_1, A_2, \dots, A_s$  be  $2n \times 2n$ -matrices, and  $A_0 = E_{2n}$ . Then  $\sum_0^s x_j A_j$  is symplectic up to the factor  $\sum_0^s x_j^2$  for all  $x_0, x_1, \dots, x_s$  if and only if  $A_1, A_2, \dots, A_s$  is a set of symplectic HR-matrices.*

*Proof.* 
$$\begin{aligned} \left( \sum_0^s x_j A_j^T \right) J \left( \sum_0^s x_j A_j \right) &= \sum_0^s x_j^2 A_j^T J A_j \\ &+ \sum_1^s x_0 x_j (A_j^T J + J A_j) + \sum_{j, k=1}^s x_j x_k (A_j^T J A_k + A_k^T J A_j), \quad j \neq k. \end{aligned}$$

Assume  $A_j^T J A_j = J, j = 0, \dots, s$ ; and

$$A_j^2 = -E, A_j A_k + A_k A_j = 0, \quad j, k = 1, \dots, s, j \neq k.$$

Then  $-A_j^T J = J A_j$ , and  $A_j^T J A_k + A_k^T J A_j = -J(A_j A_k + A_k A_j) = 0$ . Thus the whole expression reduces to  $\left( \sum_0^s x_j^2 \right) J$ . The argument is plainly reversible.

4.2. In the following, “symplectic” will mean unitary symplectic; i.e., we consider matrices from the compact group  $Sp(n) \subset U(2n)$ . A set of symplectic HR-matrices  $A_1, A_2, \dots, A_s$  is thus an  $\varepsilon$ -representation of  $G_s$  in  $Sp(n)$ ; we continue to call its degree  $2n$ . The notations  $v_s^{Sp}, d_s^{Sp}, D_s^{Sp}, E_s^{Sp}$  have the same meaning as before for  $U$  and for  $O$ .

All elements of  $G_s$  have square 1 or  $\varepsilon$ ; a matrix  $\in U(2n)$  of square  $\pm E$  is symplectic if and only if it is of the form  $\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$  with  $B^t = -B$ ,  $\bar{A}^T = A$  in the case of square  $E$ , and  $B^t = B$ ,  $\bar{A}^t = -A$  in the case of square  $-E$ . Symplectic representations of  $G_s$  are sums of irreducible unitary representations; if an irreducible unitary  $\varepsilon$ -representation is not (equivalent to a) symplectic, we have to add its conjugate-complex in order to obtain an irreducible symplectic  $\varepsilon$ -representation. Due to the description (2) of the  $G_s$  the following observations yield the complete list of degrees etc.

4.3. (a) The tensor product of a unitary representation  $V$  of even degree and an orthogonal representation (of any degree) is symplectic if and only if  $V$  is.

(b) Since  $Sp(1) = SU(2)$ , the irreducible unitary  $\varepsilon$ -representations (of degree 2) of  $G_2 = Q$  are symplectic.

(c) The irreducible  $\varepsilon$ -representations of  $D$  (= dihedral group of order 8) are not symplectic, but orthogonal; the same holds for  $D^j$  and  $D^j K$ ,  $K =$  Klein 4-group.

(d) The tensor product of any representation with the irreducible  $\varepsilon$ -representation (of degree 1) of  $G_1 = C$  is not symplectic.

The periodicity modulo 8,  $G_{s+8} = G_8 G_s = D^4 G_s$ , with  $d_8^O = d_8^U = 16$ , yields  $d_{s+8}^{Sp} = 16d_s^{Sp}$  and  $v_{s+8}^{Sp} = v_s^{Sp}$ . For  $s \equiv 2, 3, 4$  modulo 8 the irreducible unitary  $\varepsilon$ -representations of  $G_s$  are symplectic,  $d_s^{Sp} = d_s^U$  and  $v_s^{Sp} = v_s^U$ ; for the other  $s$  they are not, thus  $d_s^{Sp} = 2d_s^U$ . For  $s \equiv 1, 5$  modulo 8 the conjugate-complex representations are inequivalent, thus  $v_s^{Sp} = 1$ ; for  $s \equiv 0, 6, 7$  we combine two equivalent representations, thus  $v_s^{Sp} = v_s^U$ , i.e.,  $v_s^{Sp} = 1$  for  $s \equiv 0, 6$  and  $v_s^{Sp} = 2$  for  $s \equiv 7$ . The restriction arguments from  $G_{s+1}$  to  $G_s$  are as before and yield the  $E_s^{Sp}$ , which are periodic modulo 8.

We summarize the results in the following table

(6)	$s$	0	1	2	3	4	5	6	7	8	9
$v_s^{Sp}$		1	1	1	2	1	1	1	2	1	1
$d_s^{Sp}$		2	2	2	2	4	8	16	16	32	32
$D_s^{Sp}$		$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$
$E_s^{Sp}$		0	0	0	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0

4.4. Comparing with (3) one notes that  $D_s^O \cong D_{s+4}^{Sp}$  and  $E_s^O \cong E_{s+4}^{Sp}$ . The isomorphisms can be made explicit in terms of the  $\cup$ -product introduced in 2.2, as follows.

Let  $\rho_3 \in D_3^U = D_3^{Sp}$  be one of the generators,  $\rho_3 = \bar{\rho}_3$ , and  $\sigma_t \in D_t^O$  one of the generators. The product  $\rho_3 \cup \sigma_t \in D_{t+4}^U$  has degree  $2.2.d_t^O$ ; this is precisely the degree of a generator of  $D_{t+4}^{Sp}$ . We check that  $\rho_3 \cup \sigma_t$  is indeed in  $D_{t+4}^{Sp}$  and thus a generator: this is clear for  $t \equiv 0, 6, 7$ ,  $t+4 \equiv 2, 3, 4$  modulo 8 where  $D_{t+4}^{Sp} = D_{t+4}^U$ ; for  $t \equiv 1, 2, 3, 4, 5$  we know that  $\sigma_t = \rho_t + \bar{\rho}_t$ , whence  $\rho_3 \cup \sigma_t = \rho_3 \cup \rho_t + \bar{\rho}_3 \cup \bar{\rho}_t$ , i.e., it is one of the generators of  $D_{t+4}^{Sp}$ .

**THEOREM 4.1.** *The product of the generator  $\rho_3 \in E_3^U = E_3^{Sp}$  with  $E_s^O$  is an isomorphism  $E_s^O \cong E_{s+4}^{Sp}$  for all  $s \geq 0$ .*

4.5. We now consider the homomorphism  $\theta: E_s^{Sp} \rightarrow \pi_s(Sp)$ , analogous to  $\phi$  and  $\psi$  before.

Let  $A_1, A_2, \dots, A_s$  be a set of  $s$  symplectic  $2n \times 2n$  HR-matrices, and  $A_0 = E$ . Then

$$f_s(x_0, x_1, \dots, x_s) = \sum_0^s x_j A_j$$

$x = (x_0, x_1, \dots, x_s) \in \mathbf{R}^{s+1}$ ,  $\sum_0^s x_j^2 = 1$ , is symplectic. We consider  $f_s$  as a map  $S^s \rightarrow Sp$  via  $Sp(n)$ ; as in the cases  $U$  and  $O$  this yields a homomorphism  $\theta: E_s^{Sp} \rightarrow \pi_s(Sp)$ ,  $s \geq 0$ . The  $\pi_s(Sp)$  are known to be 0 or cyclic. Theorem A' can now be reformulated as follows.

**THEOREM B'.**  $\theta$  is an isomorphism  $E_s^{Sp} \rightarrow \pi_s(Sp)$ ,  $s \geq 0$ .

For  $s = 3$  this is clear: since  $E_3^{Sp} = E_3^U$  and  $\pi_3(Sp) \cong \pi_3(Sp(1)) = \pi_3(SU(2)) \cong \pi_3(U)$ ,  $c = \theta(\rho_3)$  is a generator of  $\pi_3(Sp) = \mathbf{Z}$ .

To complete the proof of Theorem B' we use, as for Theorem B, the  $\cup$ -product and results of  $K$ -theory relating  $K_{\mathbf{R}}$  with  $K_{\mathbf{H}}$ , the quaternionic or symplectic  $K$ -theory. The product  $c \cup b$ ,  $b \in \pi_s(O)$ , can be expressed in terms of linear maps  $S^3 \rightarrow Sp(1) = SU(2)$ ,  $S^s \rightarrow O(m)$ ,  $S^{s+4} \rightarrow U(4m)$ . As seen in 4.3, it lies in fact in  $Sp(2m) \subset U(4m)$  and can thus be regarded as an element of  $\pi_{s+4}(Sp)$ . The map  $c \cup -: \pi_s(O) \rightarrow \pi_{s+4}(Sp)$  corresponds, under  $\pi_s(O) \cong \tilde{K}_{\mathbf{R}}(S^{s+1})$  and  $\pi_t(Sp) \cong \tilde{K}_{\mathbf{H}}(S^{t+1})$ , to the isomorphism  $\tilde{K}_{\mathbf{R}}(S^{s+1}) \rightarrow \tilde{K}_{\mathbf{H}}(S^{s+5})$  given by the external tensor product of bundles with the generating bundles of  $\tilde{K}_{\mathbf{H}}(S^4) = \mathbf{Z}$  (see [K], p. 154). Hence  $c \cup -$  is an isomorphism  $\pi_s(O) \cong \pi_{s+4}(Sp)$ .

Moreover, since everything is described by linear maps the diagram

$$\begin{array}{ccc} E_s^O & \xrightarrow{\psi} & \pi_s(O) \\ \rho_3 \cup - \downarrow & & \downarrow c \cup - \\ E_{s+4}^{Sp} & \xrightarrow{\theta} & \pi_{s+4}(Sp) \end{array}$$

is commutative. The upper and the two vertical maps being isomorphisms, so is  $\theta$ .

## 5. LINEARIZATION

5.1. The groups  $E_s^U$  can be viewed, through the homomorphism  $\phi: E_s^U \rightarrow \pi_s(U)$  in 3.1, as “linear homotopy groups” of  $U$ . This means that we consider maps of  $S^s$  into  $U$  via some  $U(n)$  which are linear in the coordinates  $x_0, x_1, \dots, x_s$  of  $\mathbf{R}^{s+1} \supset S^s$ ; and linear nullhomotopies, i.e., extensions to  $S^{s+1} \rightarrow U(n)$  linear in  $x_0, x_1, \dots, x_{s+1}$ . It is an immediate corollary of Theorem B that these linear homotopy groups  $\pi_s^{\text{lin}}(U)$  are isomorphic to the  $\pi_s(U)$  by the obvious imbedding  $\pi_s^{\text{lin}}(U) \rightarrow \pi_s(U)$ . In other words:

Any map  $S^s \rightarrow U$  is homotopic to a linear map, and if a linear map  $S^s \rightarrow U$  is nullhomotopic then it admits a linear nullhomotopy.

Similar statements hold, of course, for  $\pi_s(O)$  and  $\pi_s(Sp)$ .

5.2. If these linearization phenomena could be established directly (by some approximation procedure) one would obtain a very transparent proof of the Bott periodicity theorems for  $\pi_s(U)$ ,  $\pi_s(O)$ , and  $\pi_s(Sp)$ , in the sense that they would be reduced to the algebraic computation of  $E_s^U$ ,  $E_s^O$ , and  $E_s^{Sp}$  as carried out here.

5.3. Linear maps  $S^s \rightarrow U$  via  $U(n)$ , etc., are given explicitly in terms of HR-matrices; thus the coefficients involve 0,  $\pm 1$ ,  $\pm i$  only. Such maps have a meaning over very general fields instead of  $\mathbf{R}$  and  $\mathbf{C}$ , and one should compare the corresponding linear homotopy groups with homotopy groups defined by means of algebraic maps.

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