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A REMARK ON MEROMORPHIC DIFFERENTIALS IN OPEN RIEMANN SURFACES

by Pascual Cutillas Ripoll

After the appearance of the well known paper [2] of Gunning and Narasimhan, wherein they proved the existence of a locally univalent holomorphic function on an open Riemann surface V, several generalizations of this theorem have been made by using slight modifications of the Gunning-Narasimhan arguments. One of them was that of Kusunoki-Sainouchi [3] who showed that the divisor and periods of a meromorphic differential on V can be prescribed; and, another was obtained by Schmieder [5] on demonstrating that there exists a holomorphic function on V with divisor and ramification divisor prescribed (provided that they are compatible in an obvious sense).

Our purpose on writing this paper is to show that in fact the Gunning-Narasimhan reasoning can also be slightly modified in order to prove something which seems to have not appeared in the literature until to now, and is a generalization of all the previously cited results. Namely, and roughly speaking, that the divisor, singular parts and periods of a meromorphic differential on V can be arbitrarily prescribed.

Following Kusunoki-Sainouchi we shall consider in V a canonical exhaustion, that is, a sequence (U_n) of relatively compact connected open subsets of V such that for every $n \in \mathbf{N}$, (1) $\bar{U}_n \subset U_{n+1}$, (2) $V - \bar{U}_n$ has no relatively compact connected component, (3) U_n and $V - \bar{U}_n$ have a common boundary formed by a finite set of analytic dividing curves (i.e., each of them is a Jordan closed curve which disconnects V). We shall also consider a family $F = \{A_i, B_i, C_i : i \in \mathbf{N}\}$ of analytic closed curves in V defining a basis of the first homology group $H_1(V)$ and verifying: (4) with the standard notation for intersection numbers and denoting in the same way the anterior curves and the corresponding elements of $H^1(V)$, $A_i \times B_j = \delta_{ij}$, $A_i \times A_j = B_i \times B_j = 0$ for all $i, j \in \mathbf{N}$, (5) every C_i is a dividing curve. If in addition, F verifies also, with respect to a fixed canonical exhaustion (U_n)

of V, (6) all A_i and B_i are disjoint with $\bigcup_{n=1}^{\infty} \partial U_n$, (7) every C_i is a boundary contour of some U_n, and (8) the curves of F contained in $\bar{U}_n$ form a basis of $H_1(\bar{U}_n)$ (being $\bar{U}_n$ considered as a bordered Riemann surface), then F will be called (following also Kusunoki-Sainouchi) a *canonical homology basis in V with respect to* (U_n). From now on, we shall consider a fixed canonical exhaustion (U_n) of V and a fixed canonical homology basis F in V with respect to (U_n) such that the representing curves A_i, B_i verify further that $A_i \bigcap B_i$ consists on a unique point for every $i \in \mathbf{N}$, and $A_i \bigcap A_j = B_i \bigcap B_j = A_i \bigcap B_j = \emptyset$ for all $i, j \in \mathbf{N}$ with $i \neq j$.

Let us fix also some (arbitrary) $n \in \mathbf{N}$ and let $\{\alpha_1, ..., \alpha_r\}$ and $\{\beta_1, ..., \beta_s\}$ be the subsets of F formed by the curves contained in $\bar{U}_n$ and $\bar{U}_{n+1} - \bar{U}_n$ respectively. Let K be $\bar{U}_n \bigcup \beta_1 \bigcup ... \bigcup \beta_s$; then reasoning as in Lemma 2 in Gunning-Narasimhan [2], one sees that $V - K$ has no relatively compact connected component and so by a theorem of Bishop [1], every continuous complex function in K which is holomorphic in the interior of K can be uniformly approximated in K by holomorphic functions in V. The analogous conclusion for the compact subset $Q = \alpha_1 \bigcup ... \bigcup \alpha_r$ of V is also valid.

LEMMA 1. *Let L be a compact subset of V such that $V - L$ has no relatively compact connected component, and let $\delta = \sum_{j=1}^{\infty} m_j b_j$ be a divisor on V, with $m_j > 0$ and $b_j \in V - \partial L$ for every $j \in \mathbf{N}$. Let τ be a continuous complex function in L, holomorphic in $\mathring{L}$ (the interior of L) and with divisor $\geqslant \delta \mid_{\mathring{L}}$. Then there is a sequence of holomorphic functions in V with divisor $\geqslant \delta$ which approximates τ uniformly in L.*

Proof. Consider a holomorphic function g in V with divisor δ, and apply Bishop's theorem to the function τg^{-1} in L.

From now on we shall consider a divisor $\delta = \sum_{j=1}^{\infty} m_j b_j$ as in Lemma 1, with none of the b_j contained in any curve of F. Without loss of generality it can be also supposed that b_j does not belong to the boundary of any U_n for every $j \in \mathbf{N}$.

The proof of the following lemma is almost a repetition of that of Lemma 1 in Kusunoki-Sainouchi [3] (which, in turn, is strongly inspired in Gunning-Narasimhan [2]). We include it for the sake of completeness.

LEMMA 2. *Let ω be a meromorphic differential on V having no pole in any curve of F. Then, for every $\varepsilon > 0$ and $\mu_1, ..., \mu_s \in \mathbf{C}$, there exists a* *holomorphic function* f *on* V *such that (1)* $|f| < \varepsilon$ *in* $\bar{U}_n$, *(2) the divisor of* f *is* $\geqslant \delta$, *and (3)* $\int_{\alpha_i} e^f\omega = \int_{\alpha_i}\omega$ *for* $i = 1, ..., r$, $\int_{\beta_i} e^f\omega = \mu_i$ *for* $i = 1, ..., s$.

Proof. Let $u_1, ..., u_{r+s}$ be continuous functions in $\alpha_1, ..., \alpha_r, \beta_1, ..., \beta_s$ respectively, with mutually disjoint supports, and such that $\int_{\alpha_i} u_i\omega \neq 0$ for $i = 1, ..., r$ and $\int_{\beta_i} e^{u_{r+i}}\omega = \mu_i$, $\int_{\beta_i} u_{r+i}e^{u_{r+i}}\omega \neq 0$ for $i = 1, ..., s$; and for $i = 1, ..., r$ (resp., $i = r+1, ..., r+s$) extend each u_i to K (mantaining the notation) in such a way that it is identically zero in $K - \alpha_i$ (resp. $K - \beta_{i-r}$).

Let $\varphi_i: \mathbf{C}^{r+s} \to \mathbf{C}$ be the holomorphic function defined for $i = 1, ..., r+s$ by

$$\varphi_i(z_1, ..., z_{r+s}) = \begin{cases} \displaystyle\int_{\alpha_i} \exp\left(\sum_{l=1}^{r+s} z_l u_l\right)\omega & \text{if} \quad i \leqslant r, \\ \displaystyle\int_{\beta_{i-r}} \exp\left(\sum_{l=1}^{r+s} z_l u_l\right)\omega & \text{if} \quad i > r. \end{cases}$$

Then, for $a = (0, ..., 0, 1, ..., 1) \in \mathbf{C}^{r+s}$, with the r first components having the value 0, we have

$$\varphi_i(a) = \begin{cases} \displaystyle\int_{\alpha_i} \omega & \text{if} \quad i \leqslant r, \\ \displaystyle\int_{\beta_{i-r}} e^{u_i}\omega = \mu_{i-r} & \text{if} \quad i > r, \end{cases}$$

$$\frac{\partial\varphi_i}{\partial z_i}(a) = \begin{cases} \displaystyle\int_{\alpha_i} u_i\omega \neq 0 & \text{if} \quad i \leqslant r, \\ \displaystyle\int_{\beta_{i-r}} u_i e^{u_i}\omega \neq 0 & \text{if} \quad i > r, \end{cases}$$

$$\frac{\partial\varphi_i}{\partial z_j}(a) = 0 \qquad \text{if} \quad i \neq j.$$

Let $\varphi = (\varphi_1, ..., \varphi_{r+s}): \mathbf{C}^{r+s} \to \mathbf{C}^{r+s}$. Then $\varphi(a) = (\int_{\alpha_1}\omega, ..., \int_{\alpha_r}\omega, \mu_1, ..., \mu_s)$, and it is clear that the jacobian (determinant) of φ at a is not zero. Now, Lemma 1 shows that for every $i = 1, ..., r$ there exists a sequence $\{f_{i,m}\}_{m\in\mathbf{N}}$ of holomorphic functions on V with divisor $\geqslant \delta$ (notations as just before Lemma 1) which aproximates u_i uniformly in Q; and so, there exist terms $f_1, ..., f_r$ of the sequences $\{f_{1,m}\}, ..., \{f_{r,m}\}$ respectively, such that on setting

$$\Phi_i(z_1, ..., z_{r+s}) = \begin{cases} \int_{\alpha_i} \exp\left(\sum_{l=1}^{r} z_l f_l + \sum_{l=1}^{s} z_{r+l} u_{r+l}\right)\omega & \text{if } 1 \leqslant i \leqslant r, \\ \int_{\beta_{i-r}} \exp\left(\sum_{l=1}^{r} z_l f_l + \sum_{l=1}^{s} z_{r+l} u_{r+l}\right)\omega & \text{if } r < i \leqslant r+s, \end{cases}$$

and $\Phi = (\Phi_1, ..., \Phi_{r+s})\colon \mathbf{C}^{r+s} \to \mathbf{C}^{r+s}$, then $\Phi(a) = \varphi(a)$ and the jacobian of Φ at a is not zero. Let, for $l = 1, ..., s$, $\{g_{l,m}\}_{m\in\mathbf{N}}$ be a sequence of holomorphic functions on V, with divisor $\geqslant \delta$, which aproximates u_{r+l} uniformly in K; and let

$$\psi_{i,m}(z) = \begin{cases} \int_{\alpha_i} \exp\left(\sum_{l=1}^{r} z_l f_l + \sum_{l=1}^{s} z_{r+l} g_{l,m}\right)\omega & \text{if } 1 \leqslant i \leqslant r, \\ \int_{\beta_{i-r}} \exp\left(\sum_{l=1}^{r} z_l f_l + \sum_{l=1}^{s} z_{r+l} g_{l,m}\right)\omega & \text{if } r < i \leqslant r+s, \end{cases}$$

and $\psi_m = (\psi_{1,m}, ..., \psi_{r+s,m})\colon \mathbf{C}^{r+s} \to \mathbf{C}^{r+s}$. Then, all ψ_m are holomorphic, and the sequence (ψ_m) converges uniformly to Φ on every compact subset of $\mathbf{C}^{r+s}$. Therefore, as the jacobian $\dfrac{\partial(\Phi_1, ..., \Phi_{r+s})}{\partial(z_1, ..., z_{r+s})}(a)$ is not zero, then for every $\gamma > 0$ there exists $m_0(\gamma) \in \mathbf{N}$ such that $m \geqslant m_0(\gamma)$ implies the existence of a point $a_m = (a_{1,m}, ..., a_{r+s,m})$, with $\| a_m - a \| < \gamma$ and such that $\psi_m(a_n) = \Phi(a)$ (see, for instance, Proposition 5 of page 79 of Narasimhan [4]) and so, since $\Phi(a) = \varphi(a)$, then taking into account that $(g_{l,m}) \to 0$ uniformly in $\bar{U}_m$ for $l = 1, ..., s$, it is easy to see that by choosing a suitable γ and putting $f = \sum_{l=1}^{r} a_{l,m} f_l + \sum_{l=1}^{s} a_{r+l,m} g_{l,m}$, with sufficiently large $m \geqslant m_0(\delta)$, we obtain a function with the required properties.

Let now $\delta_0 = \sum_{i=1}^{\infty} n_i a_i$, with $n_i \geqslant 0$, be a divisor in $V - \{b_j\}_{j\in\mathbf{N}}$. Let, for every $j \in \mathbf{N}$, z_j be a holomorphic coordinate in some open neighbourhood of b_j such that $z_j(b_j) = 0$, and let $P_j(1/z_j)$ be a polynomial in $1/z_j$ of degree m_j without independent term. Then, there exists a meromorphic differential ω on V whose divisor is $\delta_0 - \delta$ and whose "singular part" at b_j is precisely $P_j(1/z_j)\,dz_j$ (i.e., $\omega - P_j(1/z_j)\,dz_j$ has no singularity at b_j) for every $j \in \mathbf{N}$. For, we may consider an abelian differential ω_0 on V with precisely the singular parts defined by the $P_j(1/z_j)\,dz_j$ and multiply it by some meromorphic function with suitably chosen zeroes and poles in V and with "ones" of sufficiently large multiplicities at the points b_j. Such a function exists because a meromorphic function on V with zeroes and poles prescribed can be multiplied by the exponential of a suitable holomorphic function in order that the product has the desired "ones" with at least the desired multiplicities.

With the notation of the previous paragraph, we can already state the following.

THEOREM. *There exists a meromorphic differential in* V *with divisor* $\delta_0 - \delta$, *with precisely the singular parts defined at the* $\{b_j\}_{j \in \mathbf{N}}$ *by the* $P_j(1/z_j)\, dz_j$, *and with prescribed periods at the cycles of the canonical homology basis* F.

Proof. By applying an easy induction argument based on Lemma 2 to the sequence (U_n) we obtain a holomorphic function h in V with divisor $\geqslant \delta$ and such that $e^h\omega$ has the prescribed periods. Since e^h has at every b_j a "one" of multiplicity $\geqslant m_j (j \in \mathbf{N})$, we also deduce that $e^h\omega$ has the same singular parts that ω.

COROLLARY. *For a meromorphic function* f *in* V *it is possible to prescribe the divisors of* f *and* df, *provided that they are compatible (in the obvious sense), and the periods of* $d \log f$ *(being of course integral multiples of* $2\pi i$*) along curves defining any canonical homology basis of* V *(whenever these curves contain none of the zeroes or poles of* f*).*

Proof. If a meromorphic differential ω in V is chosen with only simple poles (corresponding to the zeroes and poles of f), suitable integral residues at these poles, suitable zeroes (corresponding to the zeroes of df at which f does not vanish) and the prescribed periods, it must be of the form $d \log f$, with f having all desired properties.

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