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ON THE POSITIVE LINEAR FUNCTIONALS ON THE DISC ALGEBRA

by Marco PAVONE

1. INTRODUCTION

The disc algebra $A = A(D)$ is the set of all continuous functions on the closed unit disc $D = \{|z| \leqslant 1\}$ which are analytic on $\mathrm{int}(D)$. Equivalently, A can be defined as the subalgebra of functions in $C(D)$ which can be approximated uniformly on D by polynomials in z; it can also be identified with the class of continuous functions on the unit circle with analytic Fourier series (see [1], pp. 5-6).

The disc algebra is a commutative Banach algebra with pointwise operations and with the usual supremum norm

$$\| f \| = \sup\{| f(x) |; x \in D\}, \qquad f \in A .$$

Every multiplicative linear functional on A is the evaluation homomorphism at some x in D, and the Gelfand theory for A becomes then particularly simple. For example, the Gelfand transform $f \mapsto f^{\wedge}$ on A is simply the inclusion homomorphism from A into $C(D)$ (see [1], p. 6).

On A we also have an involution $*$ given by

$$f^*(z) = f(\bar{z})^- , \qquad f \in A, z \in D$$

(where $\bar{z}$ or z^- denotes the complex conjugate of z).

A (complex) linear functional F on A is said to be positive if $F(f^*f) \geqslant 0$ for any f in A, and in such case we write $F \geqslant 0$. The following theorem, which has long been known in the literature, completely characterizes the class of positive linear functionals on A.

THEOREM. *Every positive linear functional* F *on* A *has the form*

$$F(f) = \int_{-1}^{1} f(t)d\mu(t), \qquad f \in A,$$

for some (finite) positive Borel measure μ *on* $[-1, 1]$.

The purpose of this note is to give an elementary proof of the theorem, by only using some "classical" properties of analytic functions and the Riesz representation theorem on the dual space of $C([-1, 1])$.

In several standard books on Banach algebras this result has been used as an example to illustrate, in a special situation, certain general theorems on commutative Banach algebras B with symmetric involution (an involution on B is symmetric if $(x^*)^\wedge = (x^\wedge)^-$ for any x in B).

In [4] for example (exercise 10, p. 289), the theorem above follows as an application of a theorem which characterizes the extreme points of the convex set $\{F \in B^*; F \geqslant 0, F(1) \leqslant 1\}$ (Theorem 11.33, p. 286). In [3] (example (a), p. 273), the general theorem in question is a Bochner-like representation theorem, where the space of integration is the space of symmetric maximal ideals of B (Theorem 3, p. 272. The original example is contained in [2], p. 450).

In all cases one looks at the theorem above in the context of a very general theory, at the expense of the special features of the disc algebra as a space of analytic functions.

2. Proof of the theorem

Let F be a positive linear functional on A. Then F is automatically norm-continuous, since A is a unital Banach algebra and $\| x^* \| = \| x \|$ for any x in A (see [4], Theorem 11.31, p. 284). We might as well include continuity as part of the definition of a positive linear functional, in order to have our proof remain elementary and independent of the general theory.

Our first step is to show that any polynomial $P(z)$ such that $P(t) > 0$ for $-1 \leqslant t \leqslant 1$ can be written as $P = f^*f$ for some f in A. We start by factoring P as

$$P(z) = c\prod_i(z-\alpha_i)\prod_j(\beta_j-z)\prod_k(z-\gamma_k)(z-\bar{\gamma}_k),$$

where $c > 0$, $\alpha_i < -1$, $\beta_j > 1$ and the γ_k's are non-real.

Now let g be the standard branch of the square root function, i.e.

$$g(\rho e^{i\theta}) = \rho^{1/2}e^{i\theta/2}, \qquad \rho > 0, -\pi < \theta < \pi .$$

Then g is an analytic function on $\mathbf{C}\backslash\{t \in \mathbf{R}; t \leqslant 0\}$, so $(z-\alpha_i)^{1/2} = g(z-\alpha_i)$ and $(\beta_j-z)^{1/2} = g(\beta_j-z)$ are analytic functions on a neighbourhood of D for all i, j. Moreover, such functions are real-valued on $[-1, 1]$, so they are self-adjoint elements of A by Schwarz's reflection principle. If we now set

$$f(z) = c^{1/2}\prod_i (z-\alpha_i)^{1/2}\prod_j (\beta_j - z)^{1/2}\prod_k (z-\gamma_k),$$

then f is in A and $f^*f = P$, as required.

In particular, $F(P) \geqslant 0$ by the positivity of F. It then follows immediately that $F(P) \geqslant 0$ for any polynomial $P(z)$ such that $P(t) \geqslant 0$ for $-1 \leqslant t \leqslant 1$: indeed, note that by the previous case

$$\varepsilon F(1) + F(P) = F(P+\varepsilon) \geqslant 0$$

for any $\varepsilon > 0$, and let ε go to zero.

If now P is any real polynomial, we can apply the previous case to $-P + \| P \|_\infty 1$ and $P + \| P \|_\infty 1$, where

$$\| P \|_\infty = \sup \{ | P(t) |; -1 \leqslant t \leqslant 1 \},$$

and we get

$$-F(1) \| P \|_\infty \leqslant F(P) \leqslant F(1) \| P \|_\infty$$

(note that $F(1) = F(1^*1) \geqslant 0$). In particular $F(P)$ is real if P is real. It follows that $F(\mathrm{Re}\, P) = \mathrm{Re}\, F(P)$ and $| F(\mathrm{Re}\, P) | \leqslant F(1) \| \mathrm{Re}\, P \|_\infty$ for an arbitrary polynomial $P(z)$. If θ is a real number such that $| F(P) | = e^{i\theta} F(P)$, we then have

$$\begin{aligned} | F(P) | &= e^{i\theta}F(P) = F(e^{i\theta}P) = \mathrm{Re}\, F(e^{i\theta}P) = F(\mathrm{Re}\, e^{i\theta}P) \\ &\leqslant F(1) \| \mathrm{Re}\, e^{i\theta}P \|_\infty \leqslant F(1) \| e^{i\theta}P \|_\infty = F(1) \| P \|_\infty . \end{aligned}$$

By the density of the polynomials in $C[-1, 1]$, F extends to a (continuous) positive linear functional on $C[-1, 1]$ of norm $F(1)$. The Riesz representation theorem then gives us a positive Borel measure μ on $[-1, 1]$ such that

$$F(P) = \int_{-1}^{1} P(t)d\mu(t), \qquad P \text{ polynomial}.$$

Finally, by the continuity of F and of the functional $f \mapsto \int f d\mu$, together with the denseness of the polynomials in A, we get

$$F(f) = \int_{-1}^{1} f(t)d\mu(t), \qquad f \in A .$$

This completes the proof of the theorem.

Remark. There is another approach that might seem more "natural" than the one we took, but which does not prove as effective. We might think of looking at the restriction map $\alpha(f) = f \mid [-1, 1], f \in A$, and define a positive linear functional on $\alpha(A)$ by $G(\alpha(f)) = F(f)$ (G is well defined because α is one-to-one by the analytic continuation principle). If G were continuous, we would use the denseness of $\alpha(A)$ in $C[-1, 1]$ to find a positive measure on $[-1, 1]$ which represents G and therefore F. We know retrospectively that G must be continuous by the existence of such representing measure, but it is not easy to prove it.

In fact, the map $\alpha(f) \mapsto f$ is not continuous (if α^{-1} were continuous, then $\alpha(A)$ would be complete. But $\alpha(A)$ contains the polynomials, so it would be $\alpha(A) = C[-1, 1]$, which is incompatible with the existence of continuous non differentiable functions on $[-1, 1]$).

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