

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 35 (1989)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE POSITIVE LINEAR FUNCTIONALS ON THE DISC ALGEBRA
Autor: Pavone, Marco
DOI: <https://doi.org/10.5169/seals-57362>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 09.12.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

ON THE POSITIVE LINEAR FUNCTIONALS ON THE DISC ALGEBRA

by Marco PAVONE

1. INTRODUCTION

The disc algebra $A = A(D)$ is the set of all continuous functions on the closed unit disc $D = \{ |z| \leq 1 \}$ which are analytic on $\text{int}(D)$. Equivalently, A can be defined as the subalgebra of functions in $C(D)$ which can be approximated uniformly on D by polynomials in z ; it can also be identified with the class of continuous functions on the unit circle with analytic Fourier series (see [1], pp. 5-6).

The disc algebra is a commutative Banach algebra with pointwise operations and with the usual supremum norm

$$\|f\| = \sup \{ |f(x)| ; x \in D \}, \quad f \in A.$$

Every multiplicative linear functional on A is the evaluation homomorphism at some x in D , and the Gelfand theory for A becomes then particularly simple. For example, the Gelfand transform $f \mapsto \hat{f}$ on A is simply the inclusion homomorphism from A into $C(D)$ (see [1], p. 6).

On A we also have an involution $*$ given by

$$f^*(z) = \overline{f(\bar{z})}, \quad f \in A, z \in D$$

(where \bar{z} or z^- denotes the complex conjugate of z).

A (complex) linear functional F on A is said to be positive if $F(f^*f) \geq 0$ for any f in A , and in such case we write $F \geq 0$. The following theorem, which has long been known in the literature, completely characterizes the class of positive linear functionals on A .

THEOREM. *Every positive linear functional F on A has the form*

$$F(f) = \int_{-1}^1 f(t) d\mu(t), \quad f \in A,$$

for some (finite) positive Borel measure μ on $[-1, 1]$.

The purpose of this note is to give an elementary proof of the theorem, by only using some "classical" properties of analytic functions and the Riesz representation theorem on the dual space of $C([-1, 1])$.

In several standard books on Banach algebras this result has been used as an example to illustrate, in a special situation, certain general theorems on commutative Banach algebras B with symmetric involution (an involution on B is symmetric if $(x^*)^\wedge = (\hat{x})^-$ for any x in B).

In [4] for example (exercise 10, p. 289), the theorem above follows as an application of a theorem which characterizes the extreme points of the convex set $\{F \in B^*; F \geq 0, F(1) \leq 1\}$ (Theorem 11.33, p. 286). In [3] (example (a), p. 273), the general theorem in question is a Bochner-like representation theorem, where the space of integration is the space of symmetric maximal ideals of B (Theorem 3, p. 272). The original example is contained in [2], p. 450).

In all cases one looks at the theorem above in the context of a very general theory, at the expense of the special features of the disc algebra as a space of analytic functions.

2. PROOF OF THE THEOREM

Let F be a positive linear functional on A . Then F is automatically norm-continuous, since A is a unital Banach algebra and $\|x^*\| = \|x\|$ for any x in A (see [4], Theorem 11.31, p. 284). We might as well include continuity as part of the definition of a positive linear functional, in order to have our proof remain elementary and independent of the general theory.

Our first step is to show that any polynomial $P(z)$ such that $P(t) > 0$ for $-1 \leq t \leq 1$ can be written as $P = f^*f$ for some f in A . We start by factoring P as

$$P(z) = c \prod_i (z - \alpha_i) \prod_j (\beta_j - z) \prod_k (z - \gamma_k)(z - \bar{\gamma}_k),$$

where $c > 0$, $\alpha_i < -1$, $\beta_j > 1$ and the γ_k 's are non-real.

Now let g be the standard branch of the square root function, i.e.

$$g(\rho e^{i\theta}) = \rho^{1/2} e^{i\theta/2}, \quad \rho > 0, -\pi < \theta < \pi.$$

Then g is an analytic function on $\mathbf{C} \setminus \{t \in \mathbf{R}; t \leq 0\}$, so $(z - \alpha_i)^{1/2} = g(z - \alpha_i)$ and $(\beta_j - z)^{1/2} = g(\beta_j - z)$ are analytic functions on a neighbourhood of D for all i, j . Moreover, such functions are real-valued on $[-1, 1]$, so they are self-adjoint elements of A by Schwarz's reflection principle. If we now set

$$f(z) = c^{1/2} \prod_i (z - \alpha_i)^{1/2} \prod_j (\beta_j - z)^{1/2} \prod_k (z - \gamma_k),$$

then f is in A and $f^*f = P$, as required.

In particular, $F(P) \geq 0$ by the positivity of F . It then follows immediately that $F(P) \geq 0$ for any polynomial $P(z)$ such that $P(t) \geq 0$ for $-1 \leq t \leq 1$: indeed, note that by the previous case

$$\varepsilon F(1) + F(P) = F(P + \varepsilon) \geq 0$$

for any $\varepsilon > 0$, and let ε go to zero.

If now P is any real polynomial, we can apply the previous case to $-P + \|P\|_\infty 1$ and $P + \|P\|_\infty 1$, where

$$\|P\|_\infty = \sup \{|P(t)|; -1 \leq t \leq 1\},$$

and we get

$$-F(1)\|P\|_\infty \leq F(P) \leq F(1)\|P\|_\infty$$

(note that $F(1) = F(1^*1) \geq 0$). In particular $F(P)$ is real if P is real. It follows that $F(\operatorname{Re} P) = \operatorname{Re} F(P)$ and $|F(\operatorname{Re} P)| \leq F(1)\|\operatorname{Re} P\|_\infty$ for an arbitrary polynomial $P(z)$. If θ is a real number such that $|F(P)| = e^{i\theta}F(P)$, we then have

$$\begin{aligned} |F(P)| &= e^{i\theta}F(P) = F(e^{i\theta}P) = \operatorname{Re} F(e^{i\theta}P) = F(\operatorname{Re} e^{i\theta}P) \\ &\leq F(1)\|\operatorname{Re} e^{i\theta}P\|_\infty \leq F(1)\|e^{i\theta}P\|_\infty = F(1)\|P\|_\infty. \end{aligned}$$

By the density of the polynomials in $C[-1, 1]$, F extends to a (continuous) positive linear functional on $C[-1, 1]$ of norm $F(1)$. The Riesz representation theorem then gives us a positive Borel measure μ on $[-1, 1]$ such that

$$F(P) = \int_{-1}^1 P(t) d\mu(t), \quad P \text{ polynomial}.$$

Finally, by the continuity of F and of the functional $f \mapsto \int f d\mu$, together with the denseness of the polynomials in A , we get

$$F(f) = \int_{-1}^1 f(t) d\mu(t), \quad f \in A.$$

This completes the proof of the theorem.

Remark. There is another approach that might seem more “natural” than the one we took, but which does not prove as effective. We might

think of looking at the restriction map $\alpha(f) = f|[-1, 1]$, $f \in A$, and define a positive linear functional on $\alpha(A)$ by $G(\alpha(f)) = F(f)$ (G is well defined because α is one-to-one by the analytic continuation principle). If G were continuous, we would use the denseness of $\alpha(A)$ in $C[-1, 1]$ to find a positive measure on $[-1, 1]$ which represents G and therefore F . We know retrospectively that G must be continuous by the existence of such representing measure, but it is not easy to prove it.

In fact, the map $\alpha(f) \mapsto f$ is not continuous (if α^{-1} were continuous, then $\alpha(A)$ would be complete. But $\alpha(A)$ contains the polynomials, so it would be $\alpha(A) = C[-1, 1]$, which is incompatible with the existence of continuous non differentiable functions on $[-1, 1]$).

Acknowledgment. It is the author's pleasure to express his debt to Professor Chernoff for much more than just the conversations related to the contents of this note. The author was Paul Chernoff's assistant for his course on Banach Algebras and Spectral Theory in Berkeley, Fall 1986.

REFERENCES

- [1] GAMELIN, T. W. *Uniform Algebras*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1969.
- [2] GELFAND, I. and M. A. NAIMARK. Rings with involutions and their representations. *Izvestiya Akad. Nauk. SSSR, Ser. Matem.*, 12 (1948), 445-480 (Russian).
- [3] NAIMARK, M. A. *Normed Rings*. Erven P. Noordhoff, Ltd., Groningen, Netherlands, 1960 (Original Russian edition, 1955).
- [4] RUDIN, W. *Functional Analysis*. McGraw-Hill, New York, 1973.

(Reçu le 28 mars 1988)

Marco Pavone

Department of Mathematics
University of California
Berkeley, California 94720 (USA)