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every number of the form $\sum_1^{N+1} a_n/3^n$, $a_n \in \{0, 1, 2\}$ is obtained from the number $\sum_1^N a_n/3^n$ by making a choice between $a_{n+1} = 0$, $a_{n+1} = 1$ and $a_{n+1} = 2$. The crucial point is that the nature of this choice does not depend on the number and does not depend on N. In $\mathbf{F}_n$, the free group with n generators, the choice that one makes to form a word of length $N + 1$ from a word of length N is independent of either the word or N. Accordingly, the graph of $\mathbf{F}_n$ is a homogeneous tree (of degree $2n$).

Cantor sets of continued fractions

Cantor point sets play an important role in measure theory and in the theory of continued fractions. The Cantor ternary set C is a basic example of an uncountable Borel-measurable set whose measure is zero (see, for example, [5], p. 44 and 63). An important object in the theory of continued fractions is the set $F(n) = \{x \in [0, 1] : x = [0; a_1, a_2, a_3, \ldots] \text{ and } a_i \leqslant n \text{ for all } i\}$, that is, the set of continued fractions of bound n (n being any positive integer). The fact that $F(n)$ is a Cantor point set depends on the property that if

$$x = [0; a_1, \ldots, a_m, b_{m+1}, b_{m+2}, \ldots] \quad \text{and} \quad y = [0; a_1, \ldots, a_m, c_{m+1}, c_{m+2}, \ldots]$$

are in $F(n)$, then $x < y$ $(x > y)$ if $b_{m+1} < c_{m+1}$ and m is odd (m is even). In particular,

$$\min F(n) = [0; n, 1, n, 1, \ldots], \max F(n) = [0; 1, n, 1, n, \ldots]$$

and $F(n)$ can be obtained by first removing from $(0, 1)$ the open intervals

$$(0, [0; n, 1, n, 1, \ldots]) \quad \text{and} \quad ([0; 1, n, 1, n, \ldots], 1),$$

then removing the intervals

$$([0; n, n, 1, n, 1, \ldots], [0; n-1, 1, n, 1, n, \ldots]),$$
$$([0; n-1, n, 1, n, 1, \ldots], [0; n-2, 1, n, 1, n, \ldots]),$$
$$\ldots, ([0; 2, n, 1, n, 1, \ldots], [0; 1, 1, n, 1, n, \ldots]),$$

and so on (see [3], p. 971).

A theorem of M. Hall Jr. says that $F(4) + F(4) + \mathbf{Z} = \mathbf{R}$ ([3], theorem 3.1), which is the analogue of $C + C = [0, 2]$. Hall actually proves more general theorems on the nature of $L(A) + L(B)$ for arbitrary Cantor point sets $L(A)$ and $L(B)$. One of the main applications of Hall's theorem is the result that the Markoff spectrum contains every real number greater than 6 (cfr. [1], p. 454). The number 6 has successively been replaced by a best possible value, called Hall's ray (≈ 4.5), by employing a refinement of Hall's original theorem (see [2]).

The set $F(2) + F(2)$ has been used in [4] to prove the existence of certain gaps in the lower Markoff spectrum. It is the proof contained there that originally inspired our geometric construction.

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