

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 35 (1989)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE CANTOR SET AND A GEOMETRIC CONSTRUCTION  
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**Kapitel:** Cantor sets of continued fractions  
**DOI:** <https://doi.org/10.5169/seals-57361>

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every number of the form  $\sum_1^{N+1} a_n/3^n$ ,  $a_n \in \{0, 1, 2\}$  is obtained from the number  $\sum_1^N a_n/3^n$  by making a choice between  $a_{n+1} = 0$ ,  $a_{n+1} = 1$  and  $a_{n+1} = 2$ . The crucial point is that the nature of this choice does not depend on the number and does not depend on  $N$ . In  $F_n$ , the free group with  $n$  generators, the choice that one makes to form a word of length  $N + 1$  from a word of length  $N$  is independent of either the word or  $N$ . Accordingly, the graph of  $F_n$  is a homogeneous tree (of degree  $2n$ ).

### CANTOR SETS OF CONTINUED FRACTIONS

Cantor point sets play an important role in measure theory and in the theory of continued fractions. The Cantor ternary set  $C$  is a basic example of an uncountable Borel-measurable set whose measure is zero (see, for example, [5], p. 44 and 63). An important object in the theory of continued fractions is the set  $F(n) = \{x \in [0, 1] : x = [0; a_1, a_2, a_3, \dots]$  and  $a_i \leq n$  for all  $i\}$ , that is, the set of continued fractions of bound  $n$  ( $n$  being any positive integer). The fact that  $F(n)$  is a Cantor point set depends on the property that if

$$x = [0; a_1, \dots, a_m, b_{m+1}, b_{m+2}, \dots] \quad \text{and} \quad y = [0; a_1, \dots, a_m, c_{m+1}, c_{m+2}, \dots]$$

are in  $F(n)$ , then  $x < y$  ( $x > y$ ) if  $b_{m+1} < c_{m+1}$  and  $m$  is odd ( $m$  is even). In particular,

$$\min F(n) = [0; n, 1, n, 1, \dots], \quad \max F(n) = [0; 1, n, 1, n, \dots]$$

and  $F(n)$  can be obtained by first removing from  $(0, 1)$  the open intervals

$$(0, [0; n, 1, n, 1, \dots]) \quad \text{and} \quad ([0; 1, n, 1, n, \dots], 1),$$

then removing the intervals

$$\begin{aligned} &([0; n, n, 1, n, 1, \dots], [0; n-1, 1, n, 1, n, \dots]), \\ &([0; n-1, n, 1, n, 1, \dots], [0; n-2, 1, n, 1, n, \dots]), \\ &\dots, ([0; 2, n, 1, n, 1, \dots], [0; 1, 1, n, 1, n, \dots]), \end{aligned}$$

and so on (see [3], p. 971).

A theorem of M. Hall Jr. says that  $F(4) + F(4) + \mathbf{Z} = \mathbf{R}$  ([3], theorem 3.1), which is the analogue of  $C + C = [0, 2]$ . Hall actually proves more general theorems on the nature of  $L(A) + L(B)$  for arbitrary Cantor point sets  $L(A)$  and  $L(B)$ . One of the main applications of Hall's theorem is the result

that the Markoff spectrum contains every real number greater than 6 (cfr. [1], p. 454). The number 6 has successively been replaced by a best possible value, called Hall's ray ( $\approx 4.5$ ), by employing a refinement of Hall's original theorem (see [2]).

The set  $F(2) + F(2)$  has been used in [4] to prove the existence of certain gaps in the lower Markoff spectrum. It is the proof contained there that originally inspired our geometric construction.

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(Reçu le 28 mars 1988)

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