

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 35 (1989)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE CANTOR SET AND A GEOMETRIC CONSTRUCTION
Autor: Pavone, Marco
Kapitel: Cantor sets of continued fractions
DOI: <https://doi.org/10.5169/seals-57361>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

every number of the form $\sum_1^{N+1} a_n/3^n$, $a_n \in \{0, 1, 2\}$ is obtained from the number $\sum_1^N a_n/3^n$ by making a choice between $a_{n+1} = 0$, $a_{n+1} = 1$ and $a_{n+1} = 2$. The crucial point is that the nature of this choice does not depend on the number and does not depend on N . In F_n , the free group with n generators, the choice that one makes to form a word of length $N + 1$ from a word of length N is independent of either the word or N . Accordingly, the graph of F_n is a homogeneous tree (of degree $2n$).

CANTOR SETS OF CONTINUED FRACTIONS

Cantor point sets play an important role in measure theory and in the theory of continued fractions. The Cantor ternary set C is a basic example of an uncountable Borel-measurable set whose measure is zero (see, for example, [5], p. 44 and 63). An important object in the theory of continued fractions is the set $F(n) = \{x \in [0, 1] : x = [0; a_1, a_2, a_3, \dots] \text{ and } a_i \leq n \text{ for all } i\}$, that is, the set of continued fractions of bound n (n being any positive integer). The fact that $F(n)$ is a Cantor point set depends on the property that if

$$x = [0; a_1, \dots, a_m, b_{m+1}, b_{m+2}, \dots] \quad \text{and} \quad y = [0; a_1, \dots, a_m, c_{m+1}, c_{m+2}, \dots]$$

are in $F(n)$, then $x < y$ ($x > y$) if $b_{m+1} < c_{m+1}$ and m is odd (m is even). In particular,

$$\min F(n) = [0; n, 1, n, 1, \dots], \max F(n) = [0; 1, n, 1, n, \dots]$$

and $F(n)$ can be obtained by first removing from $(0, 1)$ the open intervals

$$(0, [0; n, 1, n, 1, \dots]) \quad \text{and} \quad ([0; 1, n, 1, n, \dots], 1),$$

then removing the intervals

$$\begin{aligned} &([0; n, n, 1, n, 1, \dots], [0; n-1, 1, n, 1, n, \dots]), \\ &([0; n-1, n, 1, n, 1, \dots], [0; n-2, 1, n, 1, n, \dots]), \\ &\dots, ([0; 2, n, 1, n, 1, \dots], [0; 1, 1, n, 1, n, \dots]), \end{aligned}$$

and so on (see [3], p. 971).

A theorem of M. Hall Jr. says that $F(4) + F(4) + \mathbf{Z} = \mathbf{R}$ ([3], theorem 3.1), which is the analogue of $C + C = [0, 2]$. Hall actually proves more general theorems on the nature of $L(A) + L(B)$ for arbitrary Cantor point sets $L(A)$ and $L(B)$. One of the main applications of Hall's theorem is the result

that the Markoff spectrum contains every real number greater than 6 (cfr. [1], p. 454). The number 6 has successively been replaced by a best possible value, called Hall's ray (≈ 4.5), by employing a refinement of Hall's original theorem (see [2]).

The set $F(2) + F(2)$ has been used in [4] to prove the existence of certain gaps in the lower Markoff spectrum. It is the proof contained there that originally inspired our geometric construction.

REFERENCES

- [1] CUSICK, T. W. The largest gaps in the lower Markoff spectrum. *Duke Math. J.* 41 (1974), 453-463.
- [2] FREIMAN, G. A. Diophantine approximation and the geometry of numbers (Markov's problem), Kalin. Gosud. Univ., Kalinin, 1975.
- [3] HALL, M. Jr. On the sum and product of continued fractions. *Annals of Mathematics*, vol. 48, No. 4 (1947), 966-993.
- [4] KINNEY, J. R. and T. S. PITCHER. On the lower range of Perron's modular function. *Canad. J. Math.* 21 (1969), 808-816.
- [5] ROYDEN, H. L. *Real Analysis*. 2nd edition, 1968, Collier Macmillan Publishers.

(Reçu le 28 mars 1988)

Marco Pavone

Department of Mathematics
University of California
Berkeley, CA 94720 (USA)