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THE CANTOR SET AND A GEOMETRIC CONSTRUCTION

by Marco PAVONE

INTRODUCTION

The Cantor ternary set consists of all those real numbers x in $[0, 1]$ which have a ternary expansion $x = \sum_{n=1}^{\infty} a_n/3^n$ for which a_n is never 1. Equivalently, C can be obtained in a purely geometrical fashion by first removing from $[0, 1]$ the middle third $(1/3, 2/3)$, then removing the middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$ of the remaining intervals, and so on (C will be exactly the complement of the countable union of the removed intervals). If $x = \sum_{n=1}^{\infty} a_n/3^n$ is in C, the geometric interpretation of its ternary expansion is that x is the unique point in $[0, 1]$ which is reached by first staying to the left or to the right of $(1/3, 2/3)$ if $a_1 = 0$ or $a_1 = 2$ respectively, then staying to the left or to the right of the next removed interval if $a_2 = 0$ or $a_2 = 2$ respectively, and so on. It follows from the construction that C is a nowhere dense closed subset of $[0, 1]$.

A well known property of C is that any real number in $[0, 2]$ can be written as the sum of two numbers in C. The purpose of this note is to give an elementary proof of $C + C = [0, 2]$ which only uses the geometric definition of C. A refinement of the proof shows in fact that for any k in $[0, 2]$ there exists either a finite or an uncountable number of pairs x, y from C such that $x + y = k$. We also discuss the analogy between this decomposition result and certain properties of continued fractions.

THE GEOMETRIC CONSTRUCTION

We set, as usual, $C \times C = \{(x, y) \in \mathbf{R}^2 : x, y \in C\}$. Then $C + C = [0, 2]$ can be geometrically restated as

(*) for any k in $[0, 2]$ the line $x + y = k$ intersects $C \times C$ in at least one point.