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THE CANTOR SET AND A GEOMETRIC CONSTRUCTION

by Marco Pavone

Introduction

The Cantor ternary set consists of all those real numbers x in [0, 1] which have a ternary expansion $x = \sum_{n=1}^{\infty} a_n/3^n$ for which a_n is never 1. Equivalently, C can be obtained in a purely geometrical fashion by first removing from [0, 1] the middle third (1/3, 2/3), then removing the middle thirds (1/9, 2/9) and (7/9, 8/9) of the remaining intervals, and so on (C will be exactly the complement of the countable union of the removed intervals). If $x = \sum_{n=1}^{\infty} a_n/3^n$ is in C, the geometric interpretation of its ternary expansion is that x is the unique point in [0, 1] which is reached by first staying to the left or to the right of (1/3, 2/3) if $a_1 = 0$ or $a_1 = 2$ respectively, then staying to the left or to the right of the next removed interval if $a_2 = 0$ or $a_2 = 2$ respectively, and so on. It follows from the construction that C is a nowhere dense closed subset of [0, 1].

A well known property of C is that any real number in [0, 2] can be written as the sum of two numbers in C. The purpose of this note is to give an elementary proof of C + C = [0, 2] which only uses the geometric definition of C. A refinement of the proof shows in fact that for any k in [0, 2] there exists either a finite or an uncountable number of pairs x, y from C such that x + y = k. We also discuss the analogy between this decomposition result and certain properties of continued fractions.

THE GEOMETRIC CONSTRUCTION

We set, as usual, $C \times C = \{(x, y) \in \mathbb{R}^2 : x, y \in C\}$. Then C + C = [0, 2] can be geometrically restated as

(*) for any k in [0, 2] the line x + y = k intersects $C \times C$ in at least one point.

Let's agree to call a line segment in \mathbb{R}^2 "horizontal" or "vertical" if it is parallel or perpendicular to the line y=x respectively. Consider a sequence $L_0, L_1, L_2, ...$ of continuous polygonal curves in \mathbb{R}^2 with the following properties (see fig. 1-3):

- (a) L_n is contained in $[0, 1] \times [0, 1]$ for all n, and is composed by horizontal and vertical segments only.
 - (b) The vertices of L_n belong to $C \times C$ for all n.
 - (c) The endpoints of L_n are (0, 0) and (1, 1) for all n.
- (d) Each L_n contains 3^n horizontal segments, each of which has length $2^{1/2} 3^n$.
- (e) For all n, and for any k in $\{0, 2, 3^n, 4, 3^n, ..., 2\}$ the line x + y = k contains a vertical segment of L_n .
- (f) For all n, and for any k not in $\{0, 2, 3^n, 4 \cdot 3^n, ..., 2\}$ the line x + y = k meets at most one horizontal segment of L_n .

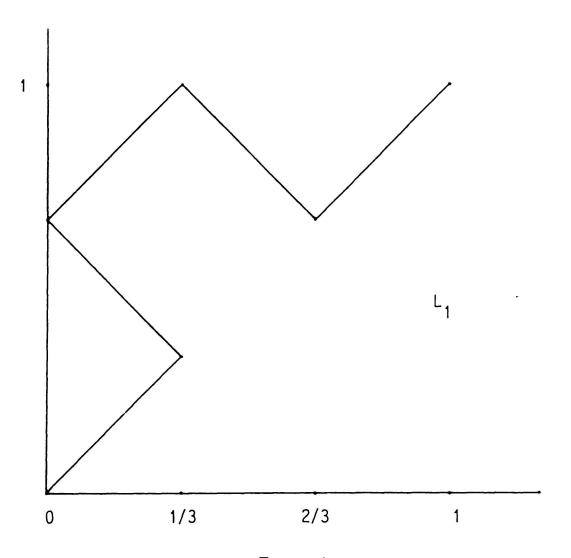
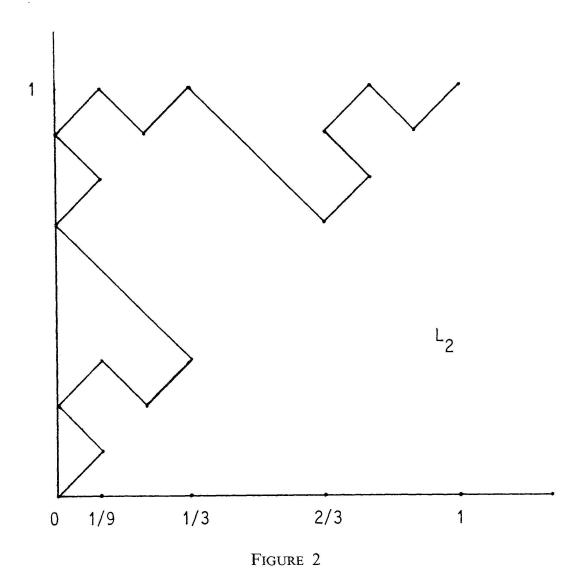


FIGURE 1

Suppose first that such a sequence exists. Then property (*) is satisfied. Indeed, fix k in [0, 2] and let r denote the line x + y = k. If k is in $\{0, 2/3^n, 4/3^n, ..., 2\}$ for some n, then r meets $C \times C$ by (e) and (b); otherwise, for any positive integer n there exists by (f) a unique horizontal segment of L_n that meets r. This implies, by (d) and (b), that dist $(r, C \times C) < 2^{1/2}/3^n$ for all positive integers n, that is, dist $(r, C \times C) = 0$. Then r meets $C \times C$ by a standard compactness argument (I recall that C is a closed subset of [0, 1]).

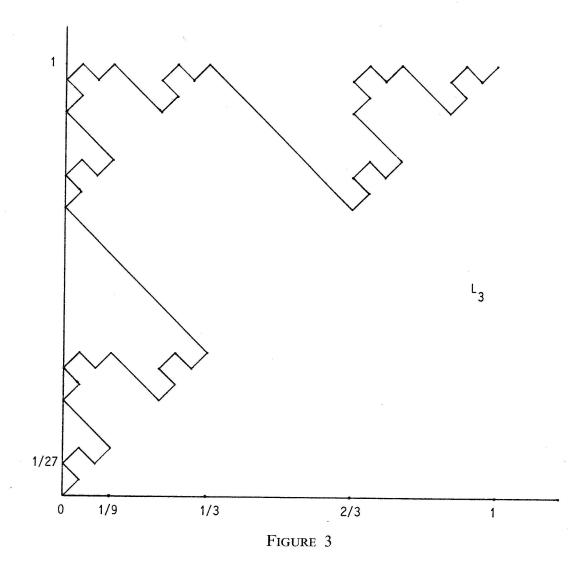


We now proceed to the heart of the argument, that is the construction of the sequence $\{L_n\}_n$. All we need is in fact the first step of an induction process. Let L_0 be the line segment with endpoints (0,0) and (1,1), and let L_1 be the polygonal with vertices (0,0), (1/3,1/3), (0,2/3), (1/3,1), (2/3,2/3) and (1,1) (see fig. 1). In general, let L_{n+1} be the curve obtained from L_n by performing on each horizontal segment of L_n the same modification that was performed on L_0 to get L_1 . In other words, we replace the generic

horizontal segment of L_n with endpoints (x, y) and $(x+1/3^n, y+1/3^n)$ by the polygonal passing through the points

$$(x, y)$$
, $(x+1/3^{n+1}, y+1/3^{n+1})$, $(x, y+2/3^{n+1})$, $(x+1/3^{n+1}, y+1/3^n)$, $(x+2/3^{n+1}, y+2/3^{n+1})$ and $(x+1/3^n, y+1/3^n)$

(see fig. 2 and 3). It is then apparent that $\{L_n\}_n$ satisfies the hypotheses (a), ..., (f) stated above.



An easy modification of the previous construction gives us more information on the way a number in [0,2] can be written as the sum of two numbers in C. For every map μ from $\mathbb{N}\setminus\{0\}$ into $\{0,2\}$ we construct a sequence $\{L_n^{(\mu)}\}_n$ of polygonal curves with properties (a), ..., (f). The idea is simply to add to the previous construction a choice between "left" and "right" at every step of the induction. What one ends up with is exactly a two-dimensional version of the geometric construction of the Cantor ternary set. We proceed as follows.

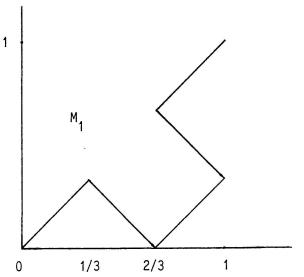


FIGURE 4

Let M_1 be the mirror image of the curve L_1 with respect to the line y=x (see figure 4). If μ is a map from $\mathbb{N}\setminus\{0\}$ into $\{0,2\}$, we define $L_0^{(\mu)}=L_0$, and for any nonnegative integer n we let $L_{n+1}^{(\mu)}$ be the polygonal obtained from $L_n^{(\mu)}$ by replacing each horizontal segment of L_n by a (normalized) copy of L_1 or M_1 , according to whether $\mu(n+1)=0$ or $\mu(n+1)=2$ respectively. For example, if $\mu=\{0,0,0,\dots\}$, we obtain our original sequence $\{L_n\}_n$ (fig. 1-3), and for $\mu=\{2,2,2,\dots\}$ we get its mirror image with respect to the line y=x. For $\mu=\{0,2,0,2,\dots\}$, we obtain castle-like polygonals as in figure 5.

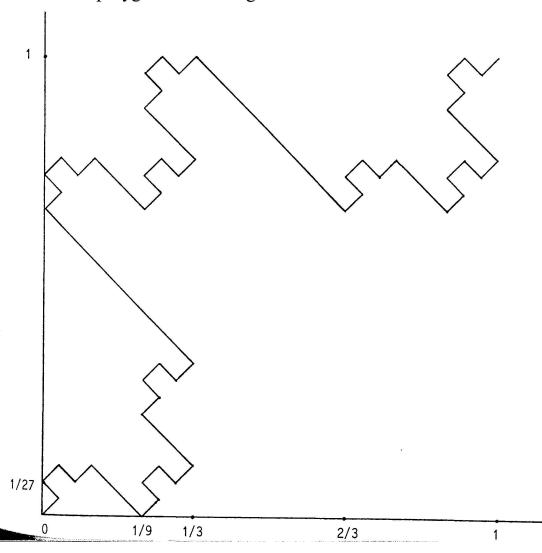


FIGURE 5

For all μ let $L^{(\mu)}$ denote the uniform limit of the curves $L_n^{(\mu)}$, n=0,1,.... Then $L^{(\mu)}$ is a continuous curve in $[0,1]\times[0,1]$ with endpoints (0,0) and (1,1), and with the property that, for any k in [0,2], the line x+y=k intersects $L^{(\mu)}$ in some point of $C\times C$. Viceversa, given any point (x,y) in $C\times C$, there is some sequence μ such that (x,y) lies on $L^{(\mu)}$.

To see this, note that the ternary subdivision of [0, 1] that generates C produces a corresponding subdivision of $[0, 1] \times [0, 1]$ that generates $C \times C$. At the *n*-th step, the subset G_n of $[0, 1] \times [0, 1]$ that contains points of $C \times C$ is the union of 4^n squares (the black squares in figure 6 for n = 3). It is clear that G_n contains the vertices of the curves $L_n^{(\mu)}$ for all μ (compare figures 3 and 6). The conclusion is now immediate.

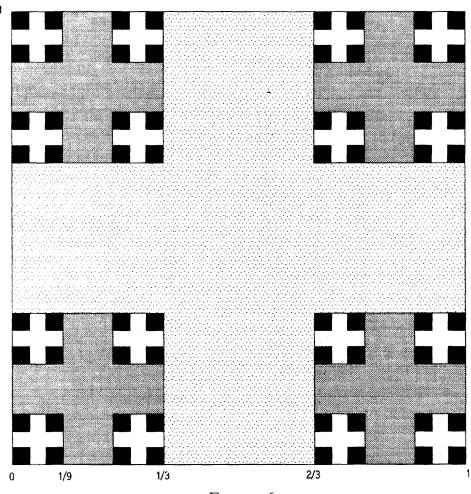


FIGURE 6

Note that if $\mu^{\hat{}}$ is the sequence obtained from μ by turning all the 0's in 2's and viceversa, then the line x + y = k intersects $L^{(\mu)}$ in a point (x, y) if and only if it intersects $L^{(\mu^{\hat{}})}$ in a point (y, x); in other words, $\mu^{\hat{}}$ does not give us any new information on the decomposition of k as a sum of numbers in C. We shall therefore restrict our attention to sequences μ with $\mu(1) = 0$ (i.e. to curves $L^{(\mu)}$ above the line y = x).

Fix k=2h in [0,2], h>0, and let $h=\sum_{n=1}^{\infty}a_n/3^n$ be the unique infinite ternary expansion of h. We claim that the equation x+y=k has a finite or an uncountable number S(k) of solutions in $C\times C$ according to whether the cardinality c(k) of the set $\{n\in \mathbb{N}\setminus\{0\}; a_n=1\}$ is finite or infinite respectively. In fact, the exact formula is S(k)=1 if c(k)=0 or 1, and $S(k)=3(2^{c(k)-2})$ otherwise.

Let r be the line x+y=k, and let n be any positive integer. With the notation set above, and with the help of figure 6, it is easy to see that $a_n=1$ if and only if G_n meets r in twice as many squares than G_{n-1} . Equivalently, $a_n=1$ if and only if, for all μ , r meets $L_{n-1}^{(\mu)}$ in the middle third of one of its horizontal segments; in other words, $a_n=1$ if and only if at the n-th step of the construction the curves $L_n^{(\mu)}$ meet r in twice as many points than the curves $L_{n-1}^{(\mu)}$. If $a_n \neq 1$, the choice between $\mu(n)=0$ and $\mu(n)=2$ at the n-th step does not produce any new intersection point. This shows that c(k) is finite or infinite depending on whether r meets the curves $L^{(\mu)}$ in a finite or an uncountable number of points, and our claim is proved.

Example. If k = 2h = 28/27 (h = 0.11122... in ternary form, with 2 repeated infinitely often), then S(k) = 6 and the possible decompositions are (in ternary form) k = 1 + 0.001, k = 0.222 + 0.002, k = 0.221 + 0.01, k = 0.21 + 0.021, k = 0.202 + 0.022 and k = 0.201 + 0.1.

In the case where c(k) is infinite, we saw that each new occurrence of 1 in the sequence $\{a_n\}_n$ produces a new choice between $\mu(n)=0$ and $\mu(n)=2$. In terms of the decomposition k=x+y, with $x=\sum_{n=1}^{\infty}b_n/3^n$ and $y=\sum_{n=1}^{\infty}c_n/3^n$, this corresponds precisely to choosing $b_n=c_n=0$ if $a_n=0$, $b_n=c_n=2$ if $a_n=2$, and finally $b_n=0$ and $c_n=2$ ($b_n=2$ and $c_n=0$) if $a_n=1$ and $\mu(n)=0$ ($\mu(n)=2$). An interesting case is k=1, that is, k=0.1111... In this case, if 1=x+y is the decomposition determined by the choice of some sequence μ , then one has precisely $x=\sum_{n=1}^{\infty}\mu(n)/3^n$.

Remark. The construction of the sequence $\{L_n\}_n$ (fig. 1-3) is similar to the ones which define by induction the continuous nowhere-differentiable function on [0, 1] or an infinite homogeneous tree with finite degree. They all provide examples of those geometric objects which are nowadays called fractals. A fractal has the property that each of its portions looks exactly like a reduced copy of the whole thing. This "homogeneousness" property has often an algebraic counterpart: in the case of the Cantor ternary set, the N-th step of its geometric construction corresponds to the fact that

every number of the form $\sum_{1}^{N+1} a_n/3^n$, $a_n \in \{0, 1, 2\}$ is obtained from the number $\sum_{1}^{N} a_n/3^n$ by making a choice between $a_{n+1} = 0$, $a_{n+1} = 1$ and $a_{n+1} = 2$. The crucial point is that the nature of this choice does not depend on the number and does not depend on N. In \mathbf{F}_n , the free group with n generators, the choice that one makes to form a word of length N+1 from a word of length N is independent of either the word or N. Accordingly, the graph of \mathbf{F}_n is a homogeneous tree (of degree 2n).

CANTOR SETS OF CONTINUED FRACTIONS

Cantor point sets play an important role in measure theory and in the theory of continued fractions. The Cantor ternary set C is a basic example of an uncountable Borel-measurable set whose measure is zero (see, for example, [5], p. 44 and 63). An important object in the theory of continued fractions is the set $F(n) = \{x \in [0, 1]: x = [0; a_1, a_2, a_3, ...] \text{ and } a_i \leq n \text{ for all } i\}$, that is, the set of continued fractions of bound n (n being any positive integer). The fact that F(n) is a Cantor point set depends on the property that if

 $x = [0; a_1, ..., a_m, b_{m+1}, b_{m+2}, ...]$ and $y = [0; a_1, ..., a_m, c_{m+1}, c_{m+2}, ...]$ are in F(n), then x < y (x > y) if $b_{m+1} < c_{m+1}$ and m is odd (m is even). In particular,

$$\min F(n) = [0; n, 1, n, 1, ...], \max F(n) = [0; 1, n, 1, n, ...]$$

and F(n) can be obtained by first removing from (0, 1) the open intervals

$$(0, [0; n, 1, n, 1, ...])$$
 and $([0; 1, n, 1, n, ...], 1)$,

then removing the intervals

$$([0; n, n, 1, n, 1, ...], [0; n-1, 1, n, 1, n, ...]),$$

 $([0; n-1, n, 1, n, 1, ...], [0; n-2, 1, n, 1, n, ...]),$
 $..., ([0; 2, n, 1, n, 1, ...], [0; 1, 1, n, 1, n, ...]),$

and so on (see [3], p. 971).

A theorem of M. Hall Jr. says that $F(4) + F(4) + \mathbf{Z} = \mathbf{R}$ ([3], theorem 3.1), which is the analogue of C + C = [0, 2]. Hall actually proves more general theorems on the nature of L(A) + L(B) for arbitrary Cantor point sets L(A) and L(B). One of the main applications of Hall's theorem is the result

that the Markoff spectrum contains every real number greater than 6 (cfr. [1], p. 454). The number 6 has successively been replaced by a best possible value, called Hall's ray (≈ 4.5), by employing a refinement of Hall's original theorem (see [2]).

The set F(2) + F(2) has been used in [4] to prove the existence of certain gaps in the lower Markoff spectrum. It is the proof contained there that originally inspired our geometric construction.

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