$$\tau(F) + F_*\tau(i_0) = \tau(j_0) + j_{0*}\tau(Id)$$

(see [Ml, Lemma 7.8]). Here F, j_0, and Id are all simple homotopy equivalences; so these Whitehead torsions vanish. Hence it follows that $\tau(i_0) = 0$, because $F_*: Wh(\pi_1(U')) \to Wh(\pi_1(E(L \times I))$ is an isomorphism. This means that U' is an s-cobordism. Therefore $(S^{n+2}, K) \in I_0(M, L)$ by Lemma 1.6. Q.E.D.

§ 5. Type 3 case

In this section we treat the case where $<m>$ or $[m]$ is of order p (p is not necessarily a prime number). We begin with

Lemma 5.1. *Suppose* $[m]$ *is of order* p. *Then if* $(S^{n+2}, K) \in I(M, L)$, *then* $(S^{n+2}, K)_p$ *is a homotopy* $(n+2)$*-sphere.*

Proof. Let r be the order of Tor $H_1(M-L; \mathbf{Z})$, and let γ be the canonical epimorphism $\pi_1(M-L) \to H_1(M-L; \mathbf{Z}) \otimes \mathbf{Z}_r$. Since the order of $\gamma(<m>)$ is p, we obtain the desired result by an argument similar to the proof of Lemma 2.1. Q.E.D.

If $p \geqslant 2$, there are infinitely many knots (S^{n+2}, K) such that $(S^{n+2}, K)_p$ is not a homotopy $(n+2)$-sphere; so Lemma 5.1 shows that $I(M, L) \subsetneq \mathscr{K}_n$ for such (M, L).

The rest of this section is devoted to looking for a non-trivial knot in $I(M, L)$ or $I_0(M, L)$. We will extend Proposition 3.6 and 4.2 to the case where $<m>$ is of order p. Lemma 5.1 reminds us of counterexamples to the generalized Smith conjecture.

Let (S^{n+2}, K) be an n-knot which bounds a disk pair (D^{n+3}, D) such that $(D^{n+3}, D)_p$ is a homotopy $(n+3)$-disk. Since $(S^{n+2}, K)_p$ is the boundary of $(D^{n+3}, D)_p$, $(S^{n+2}, K)_p$ is a homotopy $(n+2)$-sphere. If $n + 3 \geqslant 5$, then $(D^{n+3}, D)_p$ is diffeomorphic to D^{n+3} and hence $(S^{n+2}, K)_p$ is diffeomorphic to S^{n+2}.

The p-fold branched cyclic covering $(D^{n+3}, D)_p$ supports a $\mathbf{Z}_p$-action with the branch set D as the fixed point set. Let $E(D)_p$ be the exterior of D in $(D^{n+3}, D)_p$ and let $\rho: S^1 \to E(D)_p$ be an equivariant embedding of a meridian of D in $E(D)_p$, where the standard free $\mathbf{Z}_p$-action is considered on S^1. Since ρ is a homology equivalence and equivariant, the Whitehead torsion of ρ is defined in $Wh(\mathbf{Z}_p)$. Clearly it is independent of the choice of ρ; so we shall denote it by $\tau_p(D^{n+3}, D)$.

The following theorem is an extension of Proposition 3.6.

THEOREM 5.2. *Suppose* $<m>$ *is of order* p *(p may be equal to* 1*) for* (M^{n+2}, L^n) *and* $n \geqslant 4$. *Then* $(S^{n+2}, K) \in I_0(M, L)$ *if it bounds a disk pair* (D^{n+3}, D) *such that*

(1) $(D^{n+3}, D)_p$ *is diffeomorphic to* D^{n+3},

(2) $\mu_* \tau_p(D^{n+3}, D) = 0$,

where $\mu_* : Wh(\mathbf{Z}_p) \to Wh(\pi_1(M-L))$ *is the homomorphism induced from a homomorphism* $\mu : \mathbf{Z}_p \to \pi_1(M-L)$ *sending a generator of* $\mathbf{Z}_p$ *to* $<m> \in \pi_1(M-L)$.

Remark 5.3. (1) For each p, there are infinitely many n-knots satisfying the conditions (1) and (2) in Theorem 5.2. For example the $\mathbf{Z}_p$-orbit spaces of Sumners' knots [R, p. 347] (which are counterexamples to the generalized Smith conjecture) are the desired knots. In fact, $\tau_p(D^{n+3}, D) = 0$ for them.

(2) If $p = 1, 2, 3, 4$, or 6, then $Wh(\mathbf{Z}_p) = 0$. Hence the condition (2) of Theorem 5.2 is trivially satisfied in these cases.

Proof of Theorem 5.2. We shall observe that the proof of Proposition 3.6 works with a little modification. As before $E(L \times I \natural D)$ can be viewed as a cobordism relative boundary between $E(L)$ and $E(L \# K)$. We shall check that this is an s-cobordism.

The condition (1) implies that

$$\pi_1(E(D))/<m^p> \simeq \mathbf{Z}_p \tag{5.4}$$

where a meridian of D in D^{n+3} is also denoted by m. Hence it follows from the decomposition (3.7) that

$$\begin{aligned}
\pi_1(E(L \times I \natural D)) &\simeq \pi_1(E(L \times I)) \underset{<m>}{*} \pi_1(E(D)) \\
&\simeq \pi_1(E(L \times I)) \underset{\mathbf{Z}_p}{*} \pi_1(E(D))/<m^p> \\
&\qquad (\text{as } <m> \text{ is of order } p \text{ in } \pi_1(E(L \times I))) \\
&\simeq \pi_1(E(L \times I)) \qquad\qquad (\text{by } (5.4))
\end{aligned} \tag{5.5}$$

This implies that the inclusion map $i : E(L) = E(L) \times \{0\} \to E(L \times I \natural D)$ induces an isomorphism $\pi_1(E(L)) \to \pi_1(E(L \times I \natural D))$.

We consider the map $\tilde{i} : \tilde{E}(L) \to \tilde{E}(L \times I \natural D)$ lifted to the universal cover. Let $q : \tilde{E}(L \times I \natural D) \to E(L \times I \natural D)$ be the covering projection map. By (5.5) $q^{-1}(E(L \times I))$ is exactly the universal cover $\tilde{E}(L \times I)$. As for $q^{-1}(E(D))$ we need a little consideration. The above observation (5.5) shows that the image of $j_* : \pi_1(E(D)) \to \pi_1(E(L \times I \natural D))$ is isomorphic to $\mathbf{Z}_p$, where j is the inclusion map. We shall identify $j_*\pi_1(E(D))$ with $\mathbf{Z}_p$. Remember that $\mathbf{Z}_p$ acts freely on $E(D)_p$ as covering transformations.

Claim 5.6. $q^{-1}(E(D)) = E(D)_p \underset{\mathbf{Z}_p}{\times} \Pi$, where the right hand side denotes the orbit space of $E(D)_p \times \Pi$ by the diagonal $\mathbf{Z}_p$-action defined by $s \cdot (x, g) = (xs^{-1}, sg)$ for $s \in \mathbf{Z}_p$, $x \in E(D)_p$, and $g \in \Pi$.

Proof. The Π-covering $q^{-1}(E(D)) \to E(D)$ is classified by the map: $E(D) \to B\Pi$ induced from the homomorphism $j_*: \pi_1(E(D)) \to \Pi = \pi_1(E(L \times I \natural D))$. Here j_* factors through the inclusion $k: \mathbf{Z}_p \to \Pi$:

$$\begin{tikzcd} \pi_1(E(D)) \arrow[rr, "j_*"] \arrow[dr, "\ell"'] & & \Pi \\ & \mathbf{Z}_p \arrow[ur, hook, "k"'] & \end{tikzcd}$$

The pullback of the universal Π-bundle $E\Pi \to B\Pi$ by k is of the form $E\mathbf{Z}_p \underset{\mathbf{Z}_p}{\times} \Pi \to B\mathbf{Z}_p$. In fact, since $E\mathbf{Z}_p = E\Pi$, the map $(u, g) \to ug$ $(u \in E\mathbf{Z}_p, g \in \Pi)$ is defined from $E\mathbf{Z}_p \underset{\mathbf{Z}_p}{\times} \Pi$ to $E\Pi$. The map induces a Π-bundle map from $E\mathbf{Z}_p \underset{\mathbf{Z}_p}{\times} \Pi \to B\Pi$ to $E\Pi \to B\Pi$. On the other hand the covering induced from the homomorphism $\ell: \pi_1(E(D)) \to \mathbf{Z}_p$ is exactly the $\mathbf{Z}_p$-covering $E(D)_p \to E(D)$. These prove the claim.

Consequently we have a decomposition

$$(5.7) \qquad \tilde{E}(L \times I \natural D) = \tilde{E}(L \times I) \cup E(D)_p \underset{\mathbf{Z}_p}{\times} \Pi\,,$$

where $\tilde{E}(L \times I)$ and $E(D)_p \underset{\mathbf{Z}_p}{\times} \Pi$ are pasted together along $D^n \times S^1 \underset{\mathbf{Z}_p}{\times} \Pi$ equivariantly embedded in their boundaries. The condition (1) means that $E(D)_p$ is a homology circle. This together with (5.7) tells us that $\tilde{i}: \tilde{E}(L \times I) \to \tilde{E}(L \times I \natural D)$ induces an isomorphism on homology as $\mathbf{Z}[\Pi]$-modules. Hence i is a homotopy equivalence.

The decomposition (5.7) also tells us that

$$\tau(i) = \mu_* \tau_p(D^{n+3}, D) \quad \text{up to sign}\,.$$

Hence $\tau(i) = 0$ by the condition (2). Therefore $E(L \times I \natural D)$ is an s-cobordism relative boundary. The theorem then follows from Lemma 1.6. Q.E.D.

A torsion $\tau_p(S^{n+2}, K)$ is defined similarly to $\tau_p(D^{n+3}, D)$ if $(S^{n+2}, K)_p$ is a homotopy $(n+2)$-sphere. The following theorem is an extension of Proposition 4.2.

THEOREM 5.8. *Suppose* $<m>$ *is of order* p *(*p *may be equal to* 1*) for* (M^{n+2}, L^n) *and* $n \geqslant 4$. *Let* $a_{n,p} = 2$ *if* $n \equiv 0\,(4)$ *and* p *is even, and let* $a_{n,p} = 1$ *otherwise. Then* $a_{n,p}(S^{n+2}, K) \in I_0(M, L)$ *if*

(1) $\sigma(S^{n+2}, K) = 0$ *in case* n *is odd.*

(2) $(S^{n+2}, K)_p$ *is a homotopy* $(n+2)$*-sphere,*

(3) $a_{n,p}\mu_*\tau_p(S^{n+2}, K) = 0$

where μ_* *is the same as in Theorem 5.2.*

Proof. The argument developed in Steps 1, 2, and 3 of the proof of Proposition 4.2 still works. Step 4 needs a little modification. Instead of (4.10) we have

$$
\begin{array}{ccccc}
\tilde{E}(L \# K) & = & \tilde{E}(L) \cup E(K)_p & \underset{\mathbf{Z}_p}{\times} & \Pi \\
{\scriptstyle \tilde{h}_1}\downarrow & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle h_p \underset{\mathbf{Z}_p}{\times} Id} \\
\tilde{E}(L \# S^n) & = & \tilde{E}(L) \cup E(S^n)_p & \underset{\mathbf{Z}_p}{\times} & \Pi
\end{array}
\tag{5.9}
$$
,

(see (5.7)) where $h_p\colon E(K)_p \to E(S^n)_p$ denotes the lifting of h to the $\mathbf{Z}_p$-covers. Since h_p is a homology equivalence, the above diagram tells us that $\tilde{h}_1$ is a homotopy equivalence.

It also tells us that

$$\tau(h_1) = -\mu_*\tau_p(S^{n+2}, K),$$

which vanishes by the condition (3). Hence $h_1\colon E(L \# K) \to E(L \# S^n)$ is a simple homotopy equivalence.

Step 5 also needs some modification. We need to replace α and β by the canonical epimorphism $\gamma\colon \mathbf{Z} \to \mathbf{Z}_p$ and $\mu\colon \mathbf{Z}_p \to \Pi$ respectively. Then we have

$$\sigma(\bar{h}) = \mu_*\gamma_*\sigma(h).$$

Here we distinguish three cases to observe the value $\sigma(\bar{h})$.

Case 1. The case where n is odd. In this case the trivial homomorphism $\alpha\colon \mathbf{Z} \to 1$ induces an isomorphism $L_{n+3}(\mathbf{Z}, 1) \to L_{n+3}(1, 1)$ ([Wl1, 13A.8]). As observed in Step 5 of the proof of Proposition 4.2, $\alpha_*(\sigma(h))$ vanishes. Hence $\sigma(h) = 0$, so $\sigma(\bar{h}) = 0$.

Case 2. The case where $n \equiv 2\,(4)$ or p is odd. According to Wall [Wl2] or Bak [Ba], $L_{n+3}(\mathbf{Z}_p, 1) = 0$ in this case. Since $\gamma_*\sigma(h)$ lies in $L_{n+3}(\mathbf{Z}_p, 1)$, $\gamma_*\sigma(h) = 0$ and hence $\sigma(\bar{h}) = 0$.

Case 3. The case where $n \equiv 0\,(4)$ and p is even. In this case $L_{n+3}(\mathbf{Z}_p, 1) \simeq \mathbf{Z}_2$. Since the value $\gamma_* \sigma(h) \in L_{n+3}(\mathbf{Z}_p, 1)$ is additive with respect to connected sum, it necessarily vanishes for $(S^{n+2}, K) \sharp (S^{n+2}, K)$.

The rest of the argument is the same as that in Step 5. This proves the theorem. Q.E.D.

REFERENCES

[AS] ATIYAH, M. F. and I. M. SINGER. The index of elliptic operators III. *Ann. of Math. 87* (1968), 546-604.

[Ba] BAK, A. Odd dimensional surgery groups of odd torsion groups vanish. *Topology 14* (1975), 367-374.

[Br] BROWDER, W. The Kervaire invariant of framed manifolds and its generalization. *Ann. of Math. 90* (1969), 157-186.

[DF] DUNWOODY, M. J. and R. A. FENN. On the finiteness of higher dimensional knot sums. *Topology 26* (1987), 337-343.

[K] KERVAIRE, M. Les nœuds de dimension supérieure. *Bull. Soc. Math. de France 93* (1965), 225-271.

[KW] KERVAIRE, M. and C. WEBER. A survey of multidimensional knots. *Knot theory*, Lect. Notes in Math. 685, Springer, pp. 61-134, 1978.

[La] LAWSON, T. Detecting the standard embedding of $\mathbf{RP}^2$ in S^4. *Math. Ann. 267* (1984), 439-448.

[Le1] LEVINE, J. Unknotting spheres in codimension two. *Topology 4* (1965), 9-16.

[Le2] —— Knot cobordism in codimension two. *Comment. Math. Helv. 44* (1969), 229-244.

[Le3] —— Knot modules I. *Trans. A.M.S. 229* (1977), 1-50.

[Li] LITHERLAND, R. A generalization of the lightbulb theorem and *PL I*-equivalence of links. *Proc. A.M.S. 98* (1985), 353-358.

[Ma] MAEDA, T. Star decompositions along splitting groups. In preparation.

[Ms] MASUDA, M. An invariant of manifold pairs and its applications. *J. Math. Soc. of Japan.* To appear.

[Ml] MILNOR, J. W. Whitehead torsion. *Bull. A.M.S. 72* (1966), 358-426.

[MS] MILNOR, J. W. and J. D. STASHEFF. *Characteristic classes.* Ann. of Math. Studies 76, Princeton, 1974.

[My] MIYAZAKI, K. Conjugation and the prime decomposition of knots in closed, oriented 3-manifolds. Preprint.

[MB] MORGAN, J. and H. BASS. *The Smith conjecture.* Pure Appl. Math. 112, Academic Press, 1984.

[R] ROLFSEN, D. *Knots and links.* Math. Lect. Series 7, Publish or Perish Inc. 1976.

[Sc] SCHULTZ, R. Smooth structures on $S^p \times S^q$. *Ann. of Math. 90* (1969), 187-198.

[Sm1] SUMNERS, D. W. On the homology of finite cyclic coverings of higher-dimensional links. *Proc. A.M.S. 46* (1974), 143-149.

[Sm2] —— Smooth Z_p-actions on spheres which have knots pointwise fixed. *Trans. A.M.S. 205* (1975), 193-203.