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$$\tau(F) + F_*\tau(i_0) = \tau(j_0) + j_{0*}\tau(Id)$$

(see [M1, Lemma 7.8]). Here F , j_0 , and Id are all simple homotopy equivalences; so these Whitehead torsions vanish. Hence it follows that $\tau(i_0) = 0$, because $F_*: Wh(\pi_1(U')) \rightarrow Wh(\pi_1(E(L \times I)))$ is an isomorphism. This means that U' is an s -cobordism. Therefore $(S^{n+2}, K) \in I_0(M, L)$ by Lemma 1.6. Q.E.D.

§ 5. TYPE 3 CASE

In this section we treat the case where $\langle m \rangle$ or $[m]$ is of order p (p is not necessarily a prime number). We begin with

LEMMA 5.1. *Suppose $[m]$ is of order p . Then if $(S^{n+2}, K) \in I(M, L)$, then $(S^{n+2}, K)_p$ is a homotopy $(n+2)$ -sphere.*

Proof. Let r be the order of $\text{Tor } H_1(M-L; \mathbf{Z})$, and let γ be the canonical epimorphism $\pi_1(M-L) \rightarrow H_1(M-L; \mathbf{Z}) \otimes \mathbf{Z}_r$. Since the order of $\gamma(\langle m \rangle)$ is p , we obtain the desired result by an argument similar to the proof of Lemma 2.1. Q.E.D.

If $p \geq 2$, there are infinitely many knots (S^{n+2}, K) such that $(S^{n+2}, K)_p$ is not a homotopy $(n+2)$ -sphere; so Lemma 5.1 shows that $I(M, L) \subsetneq \mathcal{K}_n$ for such (M, L) .

The rest of this section is devoted to looking for a non-trivial knot in $I(M, L)$ or $I_0(M, L)$. We will extend Proposition 3.6 and 4.2 to the case where $\langle m \rangle$ is of order p . Lemma 5.1 reminds us of counterexamples to the generalized Smith conjecture.

Let (S^{n+2}, K) be an n -knot which bounds a disk pair (D^{n+3}, D) such that $(D^{n+3}, D)_p$ is a homotopy $(n+3)$ -disk. Since $(S^{n+2}, K)_p$ is the boundary of $(D^{n+3}, D)_p$, $(S^{n+2}, K)_p$ is a homotopy $(n+2)$ -sphere. If $n+3 \geq 5$, then $(D^{n+3}, D)_p$ is diffeomorphic to D^{n+3} and hence $(S^{n+2}, K)_p$ is diffeomorphic to S^{n+2} .

The p -fold branched cyclic covering $(D^{n+3}, D)_p$ supports a \mathbf{Z}_p -action with the branch set D as the fixed point set. Let $E(D)_p$ be the exterior of D in $(D^{n+3}, D)_p$ and let $\rho: S^1 \rightarrow E(D)_p$ be an equivariant embedding of a meridian of D in $E(D)_p$, where the standard free \mathbf{Z}_p -action is considered on S^1 . Since ρ is a homology equivalence and equivariant, the Whitehead torsion of ρ is defined in $Wh(\mathbf{Z}_p)$. Clearly it is independent of the choice of ρ ; so we shall denote it by $\tau_p(D^{n+3}, D)$.

The following theorem is an extension of Proposition 3.6.

THEOREM 5.2. *Suppose $\langle m \rangle$ is of order p (p may be equal to 1) for (M^{n+2}, L^n) and $n \geq 4$. Then $(S^{n+2}, K) \in I_0(M, L)$ if it bounds a disk pair (D^{n+3}, D) such that*

$$(1) \quad (D^{n+3}, D)_p \text{ is diffeomorphic to } D^{n+3},$$

$$(2) \quad \mu_* \tau_p(D^{n+3}, D) = 0,$$

where $\mu_*: Wh(\mathbf{Z}_p) \rightarrow Wh(\pi_1(M-L))$ is the homomorphism induced from a homomorphism $\mu: \mathbf{Z}_p \rightarrow \pi_1(M-L)$ sending a generator of \mathbf{Z}_p to $\langle m \rangle \in \pi_1(M-L)$.

Remark 5.3. (1) For each p , there are infinitely many n -knots satisfying the conditions (1) and (2) in Theorem 5.2. For example the \mathbf{Z}_p -orbit spaces of Sumners' knots [R, p. 347] (which are counterexamples to the generalized Smith conjecture) are the desired knots. In fact, $\tau_p(D^{n+3}, D) = 0$ for them.

(2) If $p = 1, 2, 3, 4$, or 6 , then $Wh(\mathbf{Z}_p) = 0$. Hence the condition (2) of Theorem 5.2 is trivially satisfied in these cases.

Proof of Theorem 5.2. We shall observe that the proof of Proposition 3.6 works with a little modification. As before $E(L \times I \natural D)$ can be viewed as a cobordism relative boundary between $E(L)$ and $E(L \# K)$. We shall check that this is an s -cobordism.

The condition (1) implies that

$$(5.4) \quad \pi_1(E(D))/\langle m^p \rangle \simeq \mathbf{Z}_p$$

where a meridian of D in D^{n+3} is also denoted by m . Hence it follows from the decomposition (3.7) that

$$(5.5) \quad \begin{aligned} \pi_1(E(L \times I \natural D)) &\simeq \pi_1(E(L \times I)) \underset{\langle m \rangle}{*} \pi_1(E(D)) \\ &\simeq \pi_1(E(L \times I)) \underset{\mathbf{Z}_p}{*} \pi_1(E(D))/\langle m^p \rangle \\ &\quad (\text{as } \langle m \rangle \text{ is of order } p \text{ in } \pi_1(E(L \times I))) \\ &\simeq \pi_1(E(L \times I)) \quad (\text{by (5.4)}) \end{aligned}$$

This implies that the inclusion map $i: E(L) = E(L) \times \{0\} \rightarrow E(L \times I \natural D)$ induces an isomorphism $\pi_1(E(L)) \rightarrow \pi_1(E(L \times I \natural D))$.

We consider the map $\tilde{i}: \tilde{E}(L) \rightarrow \tilde{E}(L \times I \natural D)$ lifted to the universal cover. Let $q: \tilde{E}(L \times I \natural D) \rightarrow E(L \times I \natural D)$ be the covering projection map. By (5.5) $q^{-1}(E(L \times I))$ is exactly the universal cover $\tilde{E}(L \times I)$. As for $q^{-1}(E(D))$ we need a little consideration. The above observation (5.5) shows that the image of $j_*: \pi_1(E(D)) \rightarrow \pi_1(E(L \times I \natural D))$ is isomorphic to \mathbf{Z}_p , where j is the inclusion

map. We shall identify $j_*\pi_1(E(D))$ with \mathbf{Z}_p . Remember that \mathbf{Z}_p acts freely on $E(D)_p$ as covering transformations.

Claim 5.6. $q^{-1}(E(D)) = E(D)_p \times_{\mathbf{Z}_p} \Pi$, where the right hand side denotes the orbit space of $E(D)_p \times \Pi$ by the diagonal \mathbf{Z}_p -action defined by $s \cdot (x, g) = (xs^{-1}, sg)$ for $s \in \mathbf{Z}_p$, $x \in E(D)_p$, and $g \in \Pi$.

Proof. The Π -covering $q^{-1}(E(D)) \rightarrow E(D)$ is classified by the map: $E(D) \rightarrow B\Pi$ induced from the homomorphism $j_*: \pi_1(E(D)) \rightarrow \Pi = \pi_1(E(L \times I \natural D))$. Here j_* factors through the inclusion $\ell: \mathbf{Z}_p \rightarrow \Pi$:

$$\begin{array}{ccc} \pi_1(E(D)) & \xrightarrow{j_*} & \Pi \\ \ell \searrow & & \nearrow \ell \\ & \mathbf{Z}_p & \end{array}$$

The pullback of the universal Π -bundle $E\Pi \rightarrow B\Pi$ by ℓ is of the form $E\mathbf{Z}_p \times_{\mathbf{Z}_p} \Pi \rightarrow B\mathbf{Z}_p$. In fact, since $E\mathbf{Z}_p = E\Pi$, the map $(u, g) \rightarrow ug$ ($u \in E\mathbf{Z}_p$, $g \in \Pi$) is defined from $E\mathbf{Z}_p \times_{\mathbf{Z}_p} \Pi$ to $E\Pi$. The map induces a Π -bundle map from $E\mathbf{Z}_p \times_{\mathbf{Z}_p} \Pi \rightarrow B\Pi$ to $E\Pi \rightarrow B\Pi$. On the other hand the covering induced from the homomorphism $\ell: \pi_1(E(D)) \rightarrow \mathbf{Z}_p$ is exactly the \mathbf{Z}_p -covering $E(D)_p \rightarrow E(D)$. These prove the claim.

Consequently we have a decomposition

$$(5.7) \quad \tilde{E}(L \times I \natural D) = \tilde{E}(L \times I) \cup E(D)_p \times_{\mathbf{Z}_p} \Pi,$$

where $\tilde{E}(L \times I)$ and $E(D)_p \times_{\mathbf{Z}_p} \Pi$ are pasted together along $D^n \times S^1 \times_{\mathbf{Z}_p} \Pi$ equivariantly embedded in their boundaries. The condition (1) means that $E(D)_p$ is a homology circle. This together with (5.7) tells us that $\tilde{i}: \tilde{E}(L \times I) \rightarrow \tilde{E}(L \times I \natural D)$ induces an isomorphism on homology as $\mathbf{Z}[\Pi]$ -modules. Hence i is a homotopy equivalence.

The decomposition (5.7) also tells us that

$$\tau(i) = \mu_* \tau_p(D^{n+3}, D) \quad \text{up to sign.}$$

Hence $\tau(i) = 0$ by the condition (2). Therefore $E(L \times I \natural D)$ is an s -cobordism relative boundary. The theorem then follows from Lemma 1.6. Q.E.D.

A torsion $\tau_p(S^{n+2}, K)$ is defined similarly to $\tau_p(D^{n+3}, D)$ if $(S^{n+2}, K)_p$ is a homotopy $(n+2)$ -sphere. The following theorem is an extension of Proposition 4.2.

THEOREM 5.8. Suppose $\langle m \rangle$ is of order p (p may be equal to 1) for (M^{n+2}, L^n) and $n \geq 4$. Let $a_{n,p} = 2$ if $n \equiv 0 (4)$ and p is even, and let $a_{n,p} = 1$ otherwise. Then $a_{n,p}(S^{n+2}, K) \in I_0(M, L)$ if

- (1) $\sigma(S^{n+2}, K) = 0$ in case n is odd.
- (2) $(S^{n+2}, K)_p$ is a homotopy $(n+2)$ -sphere,
- (3) $a_{n,p} \mu_* \tau_p(S^{n+2}, K) = 0$

where μ_* is the same as in Theorem 5.2.

Proof. The argument developed in Steps 1, 2, and 3 of the proof of Proposition 4.2 still works. Step 4 needs a little modification. Instead of (4.10) we have

$$(5.9) \quad \begin{array}{ccc} \tilde{E}(L \# K) & = & \tilde{E}(L) \cup E(K)_p \times_{\mathbf{Z}_p} \Pi \\ \tilde{h}_1 \downarrow & & \downarrow Id \quad \downarrow h_p \times Id \\ \tilde{E}(L \# S^n) & = & \tilde{E}(L) \cup E(S^n)_p \times_{\mathbf{Z}_p} \Pi \end{array} ,$$

(see (5.7)) where $h_p: E(K)_p \rightarrow E(S^n)_p$ denotes the lifting of h to the \mathbf{Z}_p -covers. Since h_p is a homology equivalence, the above diagram tells us that \tilde{h}_1 is a homotopy equivalence.

It also tells us that

$$\tau(h_1) = -\mu_* \tau_p(S^{n+2}, K),$$

which vanishes by the condition (3). Hence $h_1: E(L \# K) \rightarrow E(L \# S^n)$ is a simple homotopy equivalence.

Step 5 also needs some modification. We need to replace α and β by the canonical epimorphism $\gamma: \mathbf{Z} \rightarrow \mathbf{Z}_p$ and $\mu: \mathbf{Z}_p \rightarrow \Pi$ respectively. Then we have

$$\sigma(\bar{h}) = \mu_* \gamma_* \sigma(h).$$

Here we distinguish three cases to observe the value $\sigma(\bar{h})$.

Case 1. The case where n is odd. In this case the trivial homomorphism $\alpha: \mathbf{Z} \rightarrow 1$ induces an isomorphism $L_{n+3}(\mathbf{Z}, 1) \rightarrow L_{n+3}(1, 1)$ ([W11, 13A.8]). As observed in Step 5 of the proof of Proposition 4.2, $\alpha_*(\sigma(h))$ vanishes. Hence $\sigma(h) = 0$, so $\sigma(\bar{h}) = 0$.

Case 2. The case where $n \equiv 2 (4)$ or p is odd. According to Wall [W12] or Bak [Ba], $L_{n+3}(\mathbf{Z}_p, 1) = 0$ in this case. Since $\gamma_* \sigma(h)$ lies in $L_{n+3}(\mathbf{Z}_p, 1)$, $\gamma_* \sigma(h) = 0$ and hence $\sigma(\bar{h}) = 0$.

Case 3. The case where $n \equiv 0(4)$ and p is even. In this case $L_{n+3}(\mathbf{Z}_p, 1) \simeq \mathbf{Z}_2$. Since the value $\gamma_*\sigma(h) \in L_{n+3}(\mathbf{Z}_p, 1)$ is additive with respect to connected sum, it necessarily vanishes for $(S^{n+2}, K) \# (S^{n+2}, K)$.

The rest of the argument is the same as that in Step 5. This proves the theorem. Q.E.D.

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