$$\pi_1(E(L\times I \natural D)) \simeq \pi_1(E(L\times I)) \underset{<m>}{*} \pi_1(E(D))$$
$$\simeq \pi_1(E(L\times I)) * (\pi_1(E(D))/{<m>})$$

where the latter isomorphism is because $<m> = 1$ in $\pi_1(E(L\times I))$ by the assumption. Since $\pi_1(E(D))/{<m>} \simeq \pi_1(D^{n+3}) \simeq \{1\}$, we have

$$\pi_1(E(L\times I \natural D)) \simeq \pi_1(E(L\times I)) \simeq \pi_1(E(L)) . \tag{3.8}$$

Here the inclusion map $i\colon E(L) = E(L) \times \{0\} \to E(L\times I \natural D)$ induces the isomorphism.

We shall observe that i is a simple homotopy equivalence. For that purpose we consider the lifting of i to the universal covers. Since the map $\pi_1(E(D)) \to \pi_1(E(L\times I \natural D))$ induced by the inclusion map is trivial as observed above, it follows from (3.7) that

$$\tilde{E}(L\times I \natural D) = \tilde{E}(L\times I) \cup E(D) \times \Pi \tag{3.9}$$

where $\Pi = \pi_1(E(L\times I \natural D)) = \pi_1(M-L)$ and $\tilde{E}(L\times I)$ and $E(D) \times \Pi$ are pasted together Π-equivariantly along $D^{n+1} \times S^1 \times \Pi$ embedded in their boundaries. This means that $\tilde{i}_*\colon H_q(\tilde{E}(L);\mathbf{Z}) \to H_q(\tilde{E}(L\times I \natural D);\mathbf{Z})$ is an isomorphism as $\mathbf{Z}[\Pi]$-modules. Hence $i_*\colon \pi_q(E(L)) \to \pi_q(E(L\times I \natural D))$ is an isomorphism by Namioka's theorem (see [Wl1, § 4]) and hence i is a homotopy equivalence.

The assumption $<m> = 1$ together with (3.9) tells us that the Whitehead torsion $\tau(i) \in Wh(\Pi)$ of the map i comes from an element of $Wh(1)$ through the map: $Wh(1) \to Wh(\Pi)$ induced from the inclusion $1 \to \Pi$. However $Wh(1) = 0$ and hence $\tau(i) = 0$. This shows that $E(L\times I \natural D)$ is an s-cobordism relative boundary. The proposition then follows from Lemma 1.6. Q.E.D.

Proposition 3.6 gives a complete answer to the case where n is even $\geqslant 4$. It would be interesting to ask if the same conclusion still holds in the case $n = 2$.

In the next section we will improve Proposition 3.6 when n is odd $\geqslant 5$.

§ 4. An improvement

Throughout this section we assume n is odd $\geqslant 5$. Let V^{n+1} be a Seifert surface of an n-knot K in S^{n+2}. The normal bundle to V in S^{n+2} is trivial. We give the stable normal bundle of S^{n+2} a canonical framing so that V can be viewed as a framed manifold.

Remember that $\partial V = K = S^n$. We make V contractible by framed surgery without touching the boundary. As is well known this is always possible in case $\dim V = n + 1$ is odd. But in case $n + 1$ is even, we encounter an obstruction which is detected by

$$\begin{cases} \operatorname{Sign} V \in \mathbf{Z} & \text{if} \quad n + 1 \equiv 0\,(4) \\ c(V) \in \mathbf{Z}/2\mathbf{Z} & \text{if} \quad n + 1 \equiv 2\,(4) \end{cases}$$

where $c(V)$ is the Kervaire invariant of V.

Remark 4.1. Since ∂V is diffeomorphic to S^n, $c(V) = 0$ if $n + 1$ is not of the form $2^k - 2$ ([Br]).

One can see that Seifert surfaces of K are framed cobordant relative boundary to each other. Hence the values $\operatorname{Sign} V$ and $c(V)$ are independent of the choice of V. We set

$$\sigma(S^{n+2}, K) = \begin{cases} \operatorname{Sign} V & \text{if} \quad n + 1 \equiv 0\,(4), \\ c(V) & \text{if} \quad n + 1 = 2^k - 2 \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 4.2. *Suppose* $<m> = 1$ *for* (M^{n+2}, L^n) *and* n *is odd* $\geqslant 5$. *Then* $(S^{n+2}, K) \in I_0(M, L)$ *if* $\sigma(S^{n+2}, K) = 0$. *In particular,* $I_0(M, L) = \mathscr{K}_n$ *if neither* $n + 1 \equiv 0\,(4)$ *nor* $n + 1 = 2^k - 2$ *for some* k.

Combining this with Theorem 1.1, we obtain

COROLLARY 4.3. *Suppose* $<m> = 1$ *for* (M^{n+2}, L^n) *and* $n + 1 \equiv 0\,(4)\,(n \neq 3)$. *Then* $(S^{n+2}, K) \in I_0(M, L)$ *if and only if* $\sigma(S^{n+2}, K) = 0$.

The rest of this section is devoted to the proof of Proposition 4.2. Let K be an n-knot in S^{n+2} such that $\sigma(S^{n+2}, K) = 0$. We shall construct an s-cobordism relative boundary between $E(L \quad K)$ and $E(L)$. The argument is rather more complicated than that of Proposition 3.6. We need some knowledge of surgery theory.

Step 1. Let V^{n+1} be a Seifert surface of K. Push the interior of V into the interior of D^{n+3} to make it transverse to the boundary S^{n+2} of D^{n+3}. We may assume that V is $(n-1)/2$-connected, if necessary, by doing framed surgery of V within D^{n+3}. In fact, this is the method used to prove that any n-knot is concordant to a simple knot (see [KW, Chap. IV]).

In the attempt to make V $(n+1)/2$-connected (and hence V is contractible by the Poincaré duality) by framed surgery of V within D^{n+3}, one encounters an obstruction. Namely a bunch of embedded $(n+1)/2$-spheres in V does not necessarily extend to embedded $(n+3)/2$-disks whose interior lies in $D^{n+3} - V$.

But if we do framed surgery of V at the outside of D^{n+3} without touching boundary, i.e. if we do surgery on framed embeddings

$$(S^{(n+1)/2} \times D^{(n+1)/2} \times D^2, S^{(n+1)/2} \times D^{(n+1)/2} \times \{0\}) \to (D^{n+3}, V),$$

then we can make V $(n+1)/2$-connected because the obstruction is exactly $\sigma(S^{n+2}, K)$ and it vanishes by the assumption. The ambient space is, however, not D^{n+3} any more. We denote by (W, D) the resulting framed oriented pair, where D is diffeomorphic to D^{n+1}.

Step 2. We construct a boundary preserving map h:

$$\big(W; N(D), E(D)\big) \to \big(D^{n+3}; N(D^{n+1}), E(D^{n+1})\big)$$

such that

(4.4) $\quad h|_{\partial W}: \partial W = S^{n+2} \to \partial D^{n+3} = S^{n+2}$ is a homotopy equivalence,

(4.5) $\quad h|_{N(D)}: N(D) \to N(D^{n+1})$ is a diffeomorphism,

where N denotes a closed tubular neighborhood and $D^{n+1} \subset D^{n+3}$ is standardly embedded.

Since D is diffeomorphic to D^{n+1}, there is a diffeomorphism

$$g: (D^{n+1} \times D^2, D^{n+1} \times \{0\}) \to \big(N(D), D\big).$$

Here $D^{n+1} \times D^2$ can be naturally identified with $N(D^{n+1})$; so we define

(4.6) $$h|_{N(D)} = g^{-1}$$

First we extend $h|_{\partial W \cap \partial N(D)} = h|_{\partial E(K)}$ to a map from $E(K)$ to $E(\partial D^{n+1}) = E(S^n)$. The obstruction lies in groups

$$H^{q+1}\big(E(K), \partial E(K); \pi_q(E(S^n))\big).$$

Since $E(S^n)$ is homotopy equivalent to S^1, it suffices to prove

(4.7) $$H^{q+1}\big(E(K), \partial E(K); \mathbf{Z}\big) = 0 \quad \text{for} \quad q = 0, 1.$$

On the other hand we have

$$\begin{aligned}
H^{q+1}\big(E(K), \partial E(K); \mathbf{Z}\big) &\simeq H^{q+1}\big(S^{n+2}, N(K); \mathbf{Z}\big) && \text{(by excision)}\\
&\simeq \tilde{H}^q(N(K); \mathbf{Z}) && (\text{if} \quad q+1 < n+2)\\
&\simeq \tilde{H}^q(S^n; \mathbf{Z}) &&\\
&= 0 && (\text{if} \quad q \neq n)
\end{aligned}$$

Hence (4.7) is satisfied as $n \geqslant 5$.

Consequently we can extend $h|_{N(D)}$ to a map

$$h|_{N(D)\cup\partial W}\colon (N(D)\cup\partial W, \partial W) \to (N(D^{n+1})\cup\partial D^{n+3}, \partial D^{n+3}).$$

The local degree of $h|_{\partial W}\colon \partial W \to \partial D^{n+3}$ is one because $h|_{\partial W\cap N(D)} = h|_{N(K)}\colon N(K) \to N(S^n)$ is a diffeomorphism by (4.6) and $h(E(K)) \subset E(S^n)$ by the construction. Since ∂W and ∂D^{n+3} are both S^{n+2}, $h|_{\partial W}$ is a homotopy equivalence. Hence (4.4) is satisfied. Moreover (4.5) is also satisfied by (4.6). In the sequel it suffices to extend $h|_{\partial E(D)}$ to a map from $E(D)$ to $E(D^{n+1})$. This time the obstruction lies in groups

$$H^{q+1}(E(D), \partial E(D); \pi_q(E(D^{n+1}))).$$

Since $E(D^{n+1})$ is homotopy equivalent to S^1, it suffices to prove

(4.8) $$H^{q+1}(E(D), \partial E(D); \mathbf{Z}) = 0 \quad \text{for} \quad q = 0, 1.$$

By excision we have

$$H^{q+1}(E(D), \partial E(D); \mathbf{Z}) \simeq H^{q+1}(W, N(D)\cup\partial W; \mathbf{Z}).$$

Remember that W is obtained from D^{n+3} by $(n+1)/2$-surgery. It implies that

$$\tilde{H}^i(W; \mathbf{Z}) = 0 \quad \text{if} \quad i \neq (n+1)/2 + 1.$$

In particular

$$\tilde{H}^i(W; \mathbf{Z}) = 0 \quad \text{for} \quad i \leqslant 3$$

as $n \geqslant 5$. Therefore it follows from the exact sequence of the pair $(W, N(D)\cup\partial W)$ that

$$H^{q+1}(W, N(D)\cup\partial W; \mathbf{Z}) \simeq \tilde{H}^q(N(D)\cup\partial W; \mathbf{Z}) \quad \text{for} \quad q \leqslant 2.$$

Here the Mayer-Vietoris exact sequence of the triad $(N(D)\cup\partial W; N(D), \partial W)$ shows that

$$\tilde{H}^q(N(D)\cup\partial W; \mathbf{Z}) = 0 \quad \text{for} \quad q = 0, 1,$$

because $N(D)$ is contractible, $\partial W = S^{n+2}$, and $N(D) \cap \partial W = S^n \times S^1$. Hence (4.8) is satisfied, and we have obtained the desired map h.

Step 3. Since W is framed, the framing of the stable normal bundle $\nu(W)$ of W induces a stable bundle map $b\colon \nu(W) \to \nu(D^{n+3})$ which covers h. The triple $\mathscr{B} = (W, h, b)$ is called a normal map.

The identity map $Id\colon (M, L) \times I \to (M, L) \times I$ gives a normal map where the stable bundle map is also the identity. We shall denote the normal map by $\mathscr{B}_{Id} = ((M, L)\times I, Id, Id)$. The maps h and Id are both diffeomorphisms on $N(D)$ and $N(L\times I)$ respectively; so one can do the boundary connected sum of $\mathscr{B}$ and $\mathscr{B}_{Id}$ at points of K and $L \times \{1\}$. This yields a new normal map $\mathscr{B}_{Id} \natural \mathscr{B} = (M\times I \natural W, Id \natural h, Id \natural b)$. Here we naturally identify the target space $(M, L) \times I \natural (D^{n+3}, D^{n+1})$ with $(M, L) \times I$. Since $Id \# h$ is a diffeomorphism on $N(L\times I \natural D)$, it gives a product structure on $N(L\times I \# D)$. Thus we get a cobordism $E(L\times I \natural D)$ relative boundary between $E(L \# K)$ and $E(L)$.

Step 4. $Id \natural h|_{E(L)}: E(L) \to E(L) \times \{0\}$ (the 0-level) is the identity; so it is a simple homotopy equivalence. We shall observe that $h_1 = Id \natural h|_{E(L\# K)}$: $E(L \# K) \to E(L) \times \{1\}$ (the 1-level) is also a simple homotopy equivalence.

We have a decomposition

$$E(L \# K) = E(L) \cup E(K)$$

in the same sense as (3.7). Hence, similarly to (3.8) one can see

$$\pi_1(E(L \# K)) \simeq \pi_1(E(L)) \tag{4.9}$$

where the inclusion map induces the isomorphism.

We can view $E(L) \times \{1\}$ as $E(L \# S^n)$ and we also have

$$E(L \# S^n) = E(L) \cup E(S^n)\,.$$

Then the map h_1 can be viewed as the identity on $E(L)$ and h on $E(K)$. This together with (4.9) shows that $h_{1*}: \pi_1(E(L \# K)) \to \pi_1(E(L \# S^n))$ is an isomorphism.

As before we consider the map $\tilde{h}_1: \tilde{E}(L \# K) \to \tilde{E}(L \# S^n)$ lifted to the universal covers. Since $<m> = 1$, we have a diagram

$$\begin{array}{ccccc} \tilde{E}(L \# K) & = & \tilde{E}(L) \cup E(K) & \times & \Pi \\ \downarrow{\scriptstyle \tilde{h}_1} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle h|_{E(K)} \times Id} \\ \tilde{E}(L \# S^n) & = & \tilde{E}(L) \cup E(S^n) & \times & \Pi\,, \end{array} \tag{4.10}$$

where $\Pi = \pi_1(M-L)$ as before. Since $h|_{E(K)}$ is a homology equivalence, the above diagram tells us that $\tilde{h}_{1*}: H_q(\tilde{E}(L \# K); \mathbf{Z}) \to H_q(\tilde{E}(L \# S^n); \mathbf{Z})$ is an isomorphism as $\mathbf{Z}[\Pi]$-modules. Therefore h_1 is a homotopy equivalence by the same reason as before.

The assumption $<m> = 1$ together with the above diagram tells us that $\tau(h_1) \in Wh(\Pi)$ comes from an element of $Wh(1)$. Hence $\tau(h_1) = 0$ as $Wh(1) = 0$.

Step 5. By step 4 $\bar{h} = Id \natural h|_{E(L\times I\natural D)}: E(L\times I \natural D) \to E(L\times I \natural D^{n+1}) = E(L\times I)$ is a simple homotopy equivalence on the boundary. We convert $\bar{h}$ into a simple homotopy equivalence by surgery without touching the boundary. The obstruction $\sigma(\bar{h})$ lies in an L-group $L_{n+3}(\Pi, 1)$ where 1 denotes the trivial homomorphism from Π to $\mathbf{Z}_2$ (note, since M is oriented and hence so is $E(L\times I)$, the orientation homomorphism: $\Pi = \pi_1(E(L\times I)) \to \mathbf{Z}_2$ is trivial).

We have a diagram similar to (4.10):

$$\begin{array}{ccccc} E(L\times I \natural D) & = & E(L\times I) & \cup & E(D) \\ {\scriptstyle \bar{h}}\downarrow & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle h} \\ E(L\times I \natural D^{n+1}) & = & E(L\times I) & \cup & E(D^{n+1}) \,. \end{array}$$

The surgery obstruction $\sigma(h)$ to converting h to a simple homotopy equivalence by surgery without touching the boundary lies in $L_{n+3}(\mathbf{Z}, 1)$ because $\pi_1(E(D^{n+1}))$ is isomorphic to $\mathbf{Z}$. The above diagram together with the assumption $<m> = 1$ tells us that

$$\sigma(\bar{h}) = \beta_*\alpha_*\sigma(h)$$

where $\alpha_*: L_{n+3}(\mathbf{Z}, 1) \to L_{n+3}(1, 1)$ and $\beta_*: L_{n+3}(1, 1) \to L_{n+3}(\Pi, 1)$ are the homomorphisms induced from the trivial homomorphisms $\alpha: \mathbf{Z} \to 1$ and $\beta: 1 \to \Pi$ respectively. It is well-known that

$$L_{n+3}(1, 1) \simeq \begin{cases} \mathbf{Z} & \text{if} \quad n+3 \equiv 0\,(4)\,, \\ \mathbf{Z}_2 & \text{if} \quad n+3 \equiv 2\,(4)\,. \end{cases}$$

As easily observed $\alpha_*\sigma(h)$ is given by

$$\begin{cases} \operatorname{Sign} W & \text{if} \quad n+3 \equiv 0\,(4) \\ c(W) & \text{if} \quad n+3 \equiv 2\,(4) \end{cases}$$

through the above isomorphism. Remember that W is framed cobordant to D^{n+3} relative boundary by the construction. Therefore those invariants vanish and hence $\sigma(\bar{h}) = 0$.

Consequently we have obtained a cobordism U' relative boundary between $E(L \# K)$ and $E(L)$ together with a simple homotopy equivalence $F: U' \to E(L\times I)$ which is the identity on the 0-level. Let $i_0: E(L) \to U'$ and $j_0: E(L) \to E(L\times I)$ be the inclusion maps from the 0-level to the cobordisms. Since $F \circ i_0 = j_0 \circ Id$ where $Id: E(L) \to E(L)$ denotes the identity map, we have

$$\tau(F) + F_*\tau(i_0) = \tau(j_0) + j_{0*}\tau(Id)$$

(see [Ml, Lemma 7.8]). Here F, j_0, and Id are all simple homotopy equivalences; so these Whitehead torsions vanish. Hence it follows that $\tau(i_0) = 0$, because $F_*\colon Wh(\pi_1(U')) \to Wh(\pi_1(E(L\times I))$ is an isomorphism. This means that U' is an s-cobordism. Therefore $(S^{n+2}, K) \in I_0(M, L)$ by Lemma 1.6. Q.E.D.

§ 5. TYPE 3 CASE

In this section we treat the case where $<m>$ or $[m]$ is of order p (p is not necessarily a prime number). We begin with

LEMMA 5.1. *Suppose* $[m]$ *is of order* p. *Then if* $(S^{n+2}, K) \in I(M, L)$, *then* $(S^{n+2}, K)_p$ *is a homotopy* $(n+2)$*-sphere.*

Proof. Let r be the order of Tor $H_1(M-L; \mathbf{Z})$, and let γ be the canonical epimorphism $\pi_1(M-L) \to H_1(M-L; \mathbf{Z}) \otimes \mathbf{Z}_r$. Since the order of $\gamma(<m>)$ is p, we obtain the desired result by an argument similar to the proof of Lemma 2.1. Q.E.D.

If $p \geqslant 2$, there are infinitely many knots (S^{n+2}, K) such that $(S^{n+2}, K)_p$ is not a homotopy $(n+2)$-sphere; so Lemma 5.1 shows that $I(M, L) \subsetneq \mathscr{K}_n$ for such (M, L).

The rest of this section is devoted to looking for a non-trivial knot in $I(M, L)$ or $I_0(M, L)$. We will extend Proposition 3.6 and 4.2 to the case where $<m>$ is of order p. Lemma 5.1 reminds us of counterexamples to the generalized Smith conjecture.

Let (S^{n+2}, K) be an n-knot which bounds a disk pair (D^{n+3}, D) such that $(D^{n+3}, D)_p$ is a homotopy $(n+3)$-disk. Since $(S^{n+2}, K)_p$ is the boundary of $(D^{n+3}, D)_p$, $(S^{n+2}, K)_p$ is a homotopy $(n+2)$-sphere. If $n + 3 \geqslant 5$, then $(D^{n+3}, D)_p$ is diffeomorphic to D^{n+3} and hence $(S^{n+2}, K)_p$ is diffeomorphic to S^{n+2}.

The p-fold branched cyclic covering $(D^{n+3}, D)_p$ supports a $\mathbf{Z}_p$-action with the branch set D as the fixed point set. Let $E(D)_p$ be the exterior of D in $(D^{n+3}, D)_p$ and let $\rho\colon S^1 \to E(D)_p$ be an equivariant embedding of a meridian of D in $E(D)_p$, where the standard free $\mathbf{Z}_p$-action is considered on S^1. Since ρ is a homology equivalence and equivariant, the Whitehead torsion of ρ is defined in $Wh(\mathbf{Z}_p)$. Clearly it is independent of the choice of ρ; so we shall denote it by $\tau_p(D^{n+3}, D)$.

The following theorem is an extension of Proposition 3.6.