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$$\begin{aligned}\pi_1(E(L \times I \natural D)) &\simeq \pi_1(E(L \times I)) \underset{<m>}{*} \pi_1(E(D)) \\ &\simeq \pi_1(E(L \times I)) * (\pi_1(E(D))/<m>)\end{aligned}$$

where the latter isomorphism is because $<m> = 1$ in $\pi_1(E(L \times I))$ by the assumption. Since $\pi_1(E(D))/<m> \simeq \pi_1(D^{n+3}) \simeq \{1\}$, we have

$$(3.8) \quad \pi_1(E(L \times I \natural D)) \simeq \pi_1(E(L \times I)) \simeq \pi_1(E(L)).$$

Here the inclusion map $i: E(L) = E(L) \times \{0\} \rightarrow E(L \times I \natural D)$ induces the isomorphism.

We shall observe that i is a simple homotopy equivalence. For that purpose we consider the lifting of i to the universal covers. Since the map $\pi_1(E(D)) \rightarrow \pi_1(E(L \times I \natural D))$ induced by the inclusion map is trivial as observed above, it follows from (3.7) that

$$(3.9) \quad \tilde{E}(L \times I \natural D) = \tilde{E}(L \times I) \cup E(D) \times \Pi$$

where $\Pi = \pi_1(E(L \times I \natural D)) = \pi_1(M - L)$ and $\tilde{E}(L \times I)$ and $E(D) \times \Pi$ are pasted together Π -equivariantly along $D^{n+1} \times S^1 \times \Pi$ embedded in their boundaries. This means that $\tilde{i}_*: H_q(\tilde{E}(L); \mathbf{Z}) \rightarrow H_q(\tilde{E}(L \times I \natural D); \mathbf{Z})$ is an isomorphism as $\mathbf{Z}[\Pi]$ -modules. Hence $i_*: \pi_q(E(L)) \rightarrow \pi_q(E(L \times I \natural D))$ is an isomorphism by Namioka's theorem (see [W11, § 4]) and hence i is a homotopy equivalence.

The assumption $<m> = 1$ together with (3.9) tells us that the Whitehead torsion $\tau(i) \in Wh(\Pi)$ of the map i comes from an element of $Wh(1)$ through the map: $Wh(1) \rightarrow Wh(\Pi)$ induced from the inclusion $1 \rightarrow \Pi$. However $Wh(1) = 0$ and hence $\tau(i) = 0$. This shows that $E(L \times I \natural D)$ is an s -cobordism relative boundary. The proposition then follows from Lemma 1.6. Q.E.D.

Proposition 3.6 gives a complete answer to the case where n is even ≥ 4 . It would be interesting to ask if the same conclusion still holds in the case $n = 2$.

In the next section we will improve Proposition 3.6 when n is odd ≥ 5 .

§ 4. AN IMPROVEMENT

Throughout this section we assume n is odd ≥ 5 . Let V^{n+1} be a Seifert surface of an n -knot K in S^{n+2} . The normal bundle to V in S^{n+2} is trivial. We give the stable normal bundle of S^{n+2} a canonical framing so that V can be viewed as a framed manifold.

Remember that $\partial V = K = S^n$. We make V contractible by framed surgery without touching the boundary. As is well known this is always possible in case $\dim V = n + 1$ is odd. But in case $n + 1$ is even, we encounter an obstruction which is detected by

$$\begin{cases} \text{Sign } V \in \mathbf{Z} & \text{if } n + 1 \equiv 0 \pmod{4} \\ c(V) \in \mathbf{Z}/2\mathbf{Z} & \text{if } n + 1 \equiv 2 \pmod{4} \end{cases}$$

where $c(V)$ is the Kervaire invariant of V .

Remark 4.1. Since ∂V is diffeomorphic to S^n , $c(V) = 0$ if $n + 1$ is not of the form $2^k - 2$ ([Br]).

One can see that Seifert surfaces of K are framed cobordant relative boundary to each other. Hence the values $\text{Sign } V$ and $c(V)$ are independent of the choice of V . We set

$$\sigma(S^{n+2}, K) = \begin{cases} \text{Sign } V & \text{if } n + 1 \equiv 0 \pmod{4}, \\ c(V) & \text{if } n + 1 = 2^k - 2 \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 4.2. Suppose $\langle m \rangle = 1$ for (M^{n+2}, L^n) and n is odd ≥ 5 . Then $(S^{n+2}, K) \in I_0(M, L)$ if $\sigma(S^{n+2}, K) = 0$. In particular, $I_0(M, L) = \mathcal{K}_n$ if neither $n + 1 \equiv 0 \pmod{4}$ nor $n + 1 = 2^k - 2$ for some k .

Combining this with Theorem 1.1, we obtain

COROLLARY 4.3. Suppose $\langle m \rangle = 1$ for (M^{n+2}, L^n) and $n + 1 \equiv 0 \pmod{4}$ ($n \neq 3$). Then $(S^{n+2}, K) \in I_0(M, L)$ if and only if $\sigma(S^{n+2}, K) = 0$.

The rest of this section is devoted to the proof of Proposition 4.2. Let K be an n -knot in S^{n+2} such that $\sigma(S^{n+2}, K) = 0$. We shall construct an s -cobordism relative boundary between $E(L \setminus K)$ and $E(L)$. The argument is rather more complicated than that of Proposition 3.6. We need some knowledge of surgery theory.

Step 1. Let V^{n+1} be a Seifert surface of K . Push the interior of V into the interior of D^{n+3} to make it transverse to the boundary S^{n+2} of D^{n+3} . We may assume that V is $(n-1)/2$ -connected, if necessary, by doing framed surgery of V within D^{n+3} . In fact, this is the method used to prove that any n -knot is concordant to a simple knot (see [KW, Chap. IV]).

In the attempt to make V $(n+1)/2$ -connected (and hence V is contractible by the Poincaré duality) by framed surgery of V within D^{n+3} , one encounters an obstruction. Namely a bunch of embedded $(n+1)/2$ -spheres in V does

not necessarily extend to embedded $(n+3)/2$ -disks whose interior lies in $D^{n+3} - V$.

But if we do framed surgery of V at the outside of D^{n+3} without touching boundary, i.e. if we do surgery on framed embeddings

$$(S^{(n+1)/2} \times D^{(n+1)/2} \times D^2, S^{(n+1)/2} \times D^{(n+1)/2} \times \{0\}) \rightarrow (D^{n+3}, V),$$

then we can make V $(n+1)/2$ -connected because the obstruction is exactly $\sigma(S^{n+2}, K)$ and it vanishes by the assumption. The ambient space is, however, not D^{n+3} any more. We denote by (W, D) the resulting framed oriented pair, where D is diffeomorphic to D^{n+1} .

Step 2. We construct a boundary preserving map h :

$$(W; N(D), E(D)) \rightarrow (D^{n+3}; N(D^{n+1}), E(D^{n+1}))$$

such that

$$(4.4) \quad h|_{\partial W}: \partial W = S^{n+2} \rightarrow \partial D^{n+3} = S^{n+2} \quad \text{is a homotopy equivalence,}$$

$$(4.5) \quad h|_{N(D)}: N(D) \rightarrow N(D^{n+1}) \quad \text{is a diffeomorphism,}$$

where N denotes a closed tubular neighborhood and $D^{n+1} \subset D^{n+3}$ is standardly embedded.

Since D is diffeomorphic to D^{n+1} , there is a diffeomorphism

$$g: (D^{n+1} \times D^2, D^{n+1} \times \{0\}) \rightarrow (N(D), D).$$

Here $D^{n+1} \times D^2$ can be naturally identified with $N(D^{n+1})$; so we define

$$(4.6) \quad h|_{N(D)} = g^{-1}$$

First we extend $h|_{\partial W \cap \partial N(D)} = h|_{\partial E(K)}$ to a map from $E(K)$ to $E(\partial D^{n+1}) = E(S^n)$. The obstruction lies in groups

$$H^{q+1}(E(K), \partial E(K); \pi_q(E(S^n))).$$

Since $E(S^n)$ is homotopy equivalent to S^1 , it suffices to prove

$$(4.7) \quad H^{q+1}(E(K), \partial E(K); \mathbf{Z}) = 0 \quad \text{for} \quad q = 0, 1.$$

On the other hand we have

$$\begin{aligned} H^{q+1}(E(K), \partial E(K); \mathbf{Z}) &\simeq H^{q+1}(S^{n+2}, N(K); \mathbf{Z}) && \text{(by excision)} \\ &\simeq \tilde{H}^q(N(K); \mathbf{Z}) && \text{(if } q+1 < n+2) \\ &\simeq \tilde{H}^q(S^n; \mathbf{Z}) \\ &= 0 && \text{(if } q \neq n) \end{aligned}$$

Hence (4.7) is satisfied as $n \geq 5$.

Consequently we can extend $h|_{N(D)}$ to a map

$$h|_{N(D) \cup \partial W}: (N(D) \cup \partial W, \partial W) \rightarrow (N(D^{n+1}) \cup \partial D^{n+3}, \partial D^{n+3}).$$

The local degree of $h|_{\partial W}: \partial W \rightarrow \partial D^{n+3}$ is one because $h|_{\partial W \cap N(D)} = h|_{N(K)}: N(K) \rightarrow N(S^n)$ is a diffeomorphism by (4.6) and $h(E(K)) \subset E(S^n)$ by the construction. Since ∂W and ∂D^{n+3} are both S^{n+2} , $h|_{\partial W}$ is a homotopy equivalence. Hence (4.4) is satisfied. Moreover (4.5) is also satisfied by (4.6). In the sequel it suffices to extend $h|_{\partial E(D)}$ to a map from $E(D)$ to $E(D^{n+1})$. This time the obstruction lies in groups

$$H^{q+1}(E(D), \partial E(D); \pi_q(E(D^{n+1}))).$$

Since $E(D^{n+1})$ is homotopy equivalent to S^1 , it suffices to prove

$$(4.8) \quad H^{q+1}(E(D), \partial E(D); \mathbf{Z}) = 0 \quad \text{for} \quad q = 0, 1.$$

By excision we have

$$H^{q+1}(E(D), \partial E(D); \mathbf{Z}) \simeq H^{q+1}(W, N(D) \cup \partial W; \mathbf{Z}).$$

Remember that W is obtained from D^{n+3} by $(n+1)/2$ -surgery. It implies that

$$\tilde{H}^i(W; \mathbf{Z}) = 0 \quad \text{if} \quad i \neq (n+1)/2 + 1.$$

In particular

$$\tilde{H}^i(W; \mathbf{Z}) = 0 \quad \text{for} \quad i \leq 3$$

as $n \geq 5$. Therefore it follows from the exact sequence of the pair $(W, N(D) \cup \partial W)$ that

$$H^{q+1}(W, N(D) \cup \partial W; \mathbf{Z}) \simeq \tilde{H}^q(N(D) \cup \partial W; \mathbf{Z}) \quad \text{for} \quad q \leq 2.$$

Here the Mayer-Vietoris exact sequence of the triad $(N(D) \cup \partial W; N(D), \partial W)$ shows that

$$\tilde{H}^q(N(D) \cup \partial W; \mathbf{Z}) = 0 \quad \text{for} \quad q = 0, 1,$$

because $N(D)$ is contractible, $\partial W = S^{n+2}$, and $N(D) \cap \partial W = S^n \times S^1$. Hence (4.8) is satisfied, and we have obtained the desired map h .

Step 3. Since W is framed, the framing of the stable normal bundle $\nu(W)$ of W induces a stable bundle map $b: \nu(W) \rightarrow \nu(D^{n+3})$ which covers h . The triple $\mathcal{B} = (W, h, b)$ is called a normal map.

The identity map $Id: (M, L) \times I \rightarrow (M, L) \times I$ gives a normal map where the stable bundle map is also the identity. We shall denote the normal

map by $\mathcal{B}_{Id} = ((M, L) \times I, Id, Id)$. The maps h and Id are both diffeomorphisms on $N(D)$ and $N(L \times I)$ respectively; so one can do the boundary connected sum of \mathcal{B} and \mathcal{B}_{Id} at points of K and $L \times \{1\}$. This yields a new normal map $\mathcal{B}_{Id} \sharp \mathcal{B} = (M \times I \sharp W, Id \sharp h, Id \sharp b)$. Here we naturally identify the target space $(M, L) \times I \sharp (D^{n+3}, D^{n+1})$ with $(M, L) \times I$. Since $Id \sharp h$ is a diffeomorphism on $N(L \times I \sharp D)$, it gives a product structure on $N(L \times I \sharp D)$. Thus we get a cobordism $E(L \times I \sharp D)$ relative boundary between $E(L \sharp K)$ and $E(L)$.

Step 4. $Id \sharp h|_{E(L)}: E(L) \rightarrow E(L) \times \{0\}$ (the 0-level) is the identity; so it is a simple homotopy equivalence. We shall observe that $h_1 = Id \sharp h|_{E(L \sharp K)}: E(L \sharp K) \rightarrow E(L) \times \{1\}$ (the 1-level) is also a simple homotopy equivalence.

We have a decomposition

$$E(L \sharp K) = E(L) \cup E(K)$$

in the same sense as (3.7). Hence, similarly to (3.8) one can see

$$(4.9) \quad \pi_1(E(L \sharp K)) \simeq \pi_1(E(L))$$

where the inclusion map induces the isomorphism.

We can view $E(L) \times \{1\}$ as $E(L \sharp S^n)$ and we also have

$$E(L \sharp S^n) = E(L) \cup E(S^n).$$

Then the map h_1 can be viewed as the identity on $E(L)$ and h on $E(K)$. This together with (4.9) shows that $h_{1*}: \pi_1(E(L \sharp K)) \rightarrow \pi_1(E(L \sharp S^n))$ is an isomorphism.

As before we consider the map $\tilde{h}_1: \tilde{E}(L \sharp K) \rightarrow \tilde{E}(L \sharp S^n)$ lifted to the universal covers. Since $\langle m \rangle = 1$, we have a diagram

$$(4.10) \quad \begin{array}{ccccc} \tilde{E}(L \sharp K) & = & \tilde{E}(L) \cup E(K) \times \Pi \\ \tilde{h}_1 \downarrow & & \downarrow Id & & \downarrow h|_{E(K)} \times Id \\ \tilde{E}(L \sharp S^n) & = & \tilde{E}(L) \cup E(S^n) \times \Pi, \end{array}$$

where $\Pi = \pi_1(M - L)$ as before. Since $h|_{E(K)}$ is a homology equivalence, the above diagram tells us that $\tilde{h}_{1*}: H_q(\tilde{E}(L \sharp K); \mathbf{Z}) \rightarrow H_q(\tilde{E}(L \sharp S^n); \mathbf{Z})$ is an isomorphism as $\mathbf{Z}[\Pi]$ -modules. Therefore h_1 is a homotopy equivalence by the same reason as before.

The assumption $\langle m \rangle = 1$ together with the above diagram tells us that $\tau(h_1) \in Wh(\Pi)$ comes from an element of $Wh(1)$. Hence $\tau(h_1) = 0$ as $Wh(1) = 0$.

Step 5. By step 4 $\bar{h} = Id \natural h|_{E(L \times I \natural D)}: E(L \times I \natural D) \rightarrow E(L \times I \natural D^{n+1}) = E(L \times I)$ is a simple homotopy equivalence on the boundary. We convert \bar{h} into a simple homotopy equivalence by surgery without touching the boundary. The obstruction $\sigma(\bar{h})$ lies in an L -group $L_{n+3}(\Pi, 1)$ where 1 denotes the trivial homomorphism from Π to \mathbf{Z}_2 (note, since M is oriented and hence so is $E(L \times I)$, the orientation homomorphism: $\Pi = \pi_1(E(L \times I)) \rightarrow \mathbf{Z}_2$ is trivial).

We have a diagram similar to (4.10):

$$\begin{array}{ccc} E(L \times I \natural D) & = & E(L \times I) \cup E(D) \\ \bar{h} \downarrow & & \downarrow Id \quad \downarrow h \\ E(L \times I \natural D^{n+1}) & = & E(L \times I) \cup E(D^{n+1}). \end{array}$$

The surgery obstruction $\sigma(h)$ to converting h to a simple homotopy equivalence by surgery without touching the boundary lies in $L_{n+3}(\mathbf{Z}, 1)$ because $\pi_1(E(D^{n+1}))$ is isomorphic to \mathbf{Z} . The above diagram together with the assumption $\langle m \rangle = 1$ tells us that

$$\sigma(\bar{h}) = \beta_* \alpha_* \sigma(h)$$

where $\alpha_*: L_{n+3}(\mathbf{Z}, 1) \rightarrow L_{n+3}(1, 1)$ and $\beta_*: L_{n+3}(1, 1) \rightarrow L_{n+3}(\Pi, 1)$ are the homomorphisms induced from the trivial homomorphisms $\alpha: \mathbf{Z} \rightarrow 1$ and $\beta: 1 \rightarrow \Pi$ respectively. It is well-known that

$$L_{n+3}(1, 1) \simeq \begin{cases} \mathbf{Z} & \text{if } n+3 \equiv 0 (4), \\ \mathbf{Z}_2 & \text{if } n+3 \equiv 2 (4). \end{cases}$$

As easily observed $\alpha_* \sigma(h)$ is given by

$$\begin{cases} \text{Sign } W & \text{if } n+3 \equiv 0 (4) \\ c(W) & \text{if } n+3 \equiv 2 (4) \end{cases}$$

through the above isomorphism. Remember that W is framed cobordant to D^{n+3} relative boundary by the construction. Therefore those invariants vanish and hence $\sigma(\bar{h}) = 0$.

Consequently we have obtained a cobordism U' relative boundary between $E(L \natural K)$ and $E(L)$ together with a simple homotopy equivalence $F: U' \rightarrow E(L \times I)$ which is the identity on the 0-level. Let $i_0: E(L) \rightarrow U'$ and $j_0: E(L) \rightarrow E(L \times I)$ be the inclusion maps from the 0-level to the cobordisms. Since $F \circ i_0 = j_0 \circ Id$ where $Id: E(L) \rightarrow E(L)$ denotes the identity map, we have

$$\tau(F) + F_*\tau(i_0) = \tau(j_0) + j_{0*}\tau(Id)$$

(see [M1, Lemma 7.8]). Here F , j_0 , and Id are all simple homotopy equivalences; so these Whitehead torsions vanish. Hence it follows that $\tau(i_0) = 0$, because $F_*: Wh(\pi_1(U')) \rightarrow Wh(\pi_1(E(L \times I)))$ is an isomorphism. This means that U' is an s -cobordism. Therefore $(S^{n+2}, K) \in I_0(M, L)$ by Lemma 1.6. Q.E.D.

§ 5. TYPE 3 CASE

In this section we treat the case where $\langle m \rangle$ or $[m]$ is of order p (p is not necessarily a prime number). We begin with

LEMMA 5.1. *Suppose $[m]$ is of order p . Then if $(S^{n+2}, K) \in I(M, L)$, then $(S^{n+2}, K)_p$ is a homotopy $(n+2)$ -sphere.*

Proof. Let r be the order of $\text{Tor } H_1(M-L; \mathbf{Z})$, and let γ be the canonical epimorphism $\pi_1(M-L) \rightarrow H_1(M-L; \mathbf{Z}) \otimes \mathbf{Z}_r$. Since the order of $\gamma(\langle m \rangle)$ is p , we obtain the desired result by an argument similar to the proof of Lemma 2.1. Q.E.D.

If $p \geq 2$, there are infinitely many knots (S^{n+2}, K) such that $(S^{n+2}, K)_p$ is not a homotopy $(n+2)$ -sphere; so Lemma 5.1 shows that $I(M, L) \subsetneq \mathcal{K}_n$ for such (M, L) .

The rest of this section is devoted to looking for a non-trivial knot in $I(M, L)$ or $I_0(M, L)$. We will extend Proposition 3.6 and 4.2 to the case where $\langle m \rangle$ is of order p . Lemma 5.1 reminds us of counterexamples to the generalized Smith conjecture.

Let (S^{n+2}, K) be an n -knot which bounds a disk pair (D^{n+3}, D) such that $(D^{n+3}, D)_p$ is a homotopy $(n+3)$ -disk. Since $(S^{n+2}, K)_p$ is the boundary of $(D^{n+3}, D)_p$, $(S^{n+2}, K)_p$ is a homotopy $(n+2)$ -sphere. If $n+3 \geq 5$, then $(D^{n+3}, D)_p$ is diffeomorphic to D^{n+3} and hence $(S^{n+2}, K)_p$ is diffeomorphic to S^{n+2} .

The p -fold branched cyclic covering $(D^{n+3}, D)_p$ supports a \mathbf{Z}_p -action with the branch set D as the fixed point set. Let $E(D)_p$ be the exterior of D in $(D^{n+3}, D)_p$ and let $\rho: S^1 \rightarrow E(D)_p$ be an equivariant embedding of a meridian of D in $E(D)_p$, where the standard free \mathbf{Z}_p -action is considered on S^1 . Since ρ is a homology equivalence and equivariant, the Whitehead torsion of ρ is defined in $Wh(\mathbf{Z}_p)$. Clearly it is independent of the choice of ρ ; so we shall denote it by $\tau_p(D^{n+3}, D)$.

The following theorem is an extension of Proposition 3.6.