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$$\pi_1(E(L \times I \natural D)) \simeq \pi_1(E(L \times I)) \underset{< m >}{*} \pi_1(E(D))$$

$$\simeq \pi_1(E(L \times I)) * (\pi_1(E(D))/< m >)$$

where the latter isomorphism is because < m> = 1 in  $\pi_1(E(L \times I))$  by the assumption. Since  $\pi_1(E(D))/< m> \simeq \pi_1(D^{n+3}) \simeq \{1\}$ , we have

(3.8) 
$$\pi_1(E(L \times I \nmid D)) \simeq \pi_1(E(L \times I)) \simeq \pi_1(E(L)).$$

Here the inclusion map  $i: E(L) = E(L) \times \{0\} \rightarrow E(L \times I \nmid D)$  induces the isomorphism.

We shall observe that i is a simple homotopy equivalence. For that purpose we consider the lifting of i to the universal covers. Since the map  $\pi_1(E(D)) \to \pi_1(E(L \times I \nmid D))$  induced by the inclusion map is trivial as observed above, it follows from (3.7) that

(3.9) 
$$\tilde{E}(L \times I \nmid D) = \tilde{E}(L \times I) \cup E(D) \times \Pi$$

where  $\Pi = \pi_1(E(L \times I \nmid D)) = \pi_1(M - L)$  and  $\tilde{E}(L \times I)$  and  $E(D) \times \Pi$  are pasted together  $\Pi$ -equivariantly along  $D^{n+1} \times S^1 \times \Pi$  embedded in their boundaries. This means that  $\tilde{i}_*: H_q(\tilde{E}(L); \mathbf{Z}) \to H_q(\tilde{E}(L \times I \nmid D); \mathbf{Z})$  is an isomorphism as  $\mathbf{Z}[\Pi]$ -modules. Hence  $i_*: \pi_q(E(L)) \to \pi_q(E(L \times I \nmid D))$  is an isomorphism by Namioka's theorem (see [W11, § 4]) and hence i is a homotopy equivalence.

The assumption  $\langle m \rangle = 1$  together with (3.9) tells us that the Whitehead torsion  $\tau(i) \in Wh(\Pi)$  of the map i comes from an element of Wh(1) through the map:  $Wh(1) \to Wh(\Pi)$  induced from the inclusion  $1 \to \Pi$ . However Wh(1) = 0 and hence  $\tau(i) = 0$ . This shows that  $E(L \times I \nmid D)$  is an s-cobordism relative boundary. The proposition then follows from Lemma 1.6. Q.E.D.

Proposition 3.6 gives a complete answer to the case where n is even  $\ge 4$ . It would be interesting to ask if the same conclusion still holds in the case n = 2.

In the next section we will improve Proposition 3.6 when n is odd  $\geq 5$ .

# § 4. An improvement

Throughout this section we assume n is odd  $\geq 5$ . Let  $V^{n+1}$  be a Seifert surface of an n-knot K in  $S^{n+2}$ . The normal bundle to V in  $S^{n+2}$  is trivial. We give the stable normal bundle of  $S^{n+2}$  a canonical framing so that V can be viewed as a framed manifold.

Remember that  $\partial V = K = S^n$ . We make V contractible by framed surgery without touching the boundary. As is well known this is always possible in case dim V = n + 1 is odd. But in case n + 1 is even, we encounter an obstruction which is detected by

$$\begin{cases} \text{Sign } V \in \mathbf{Z} & \text{if } n+1 \equiv 0 \text{ (4)} \\ c(V) \in \mathbf{Z}/2\mathbf{Z} & \text{if } n+1 \equiv 2 \text{ (4)} \end{cases}$$

where c(V) is the Kervaire invariant of V.

Remark 4.1. Since  $\partial V$  is diffeomorphic to  $S^n$ , c(V) = 0 if n + 1 is not of the form  $2^k - 2$  ([Br]).

One can see that Seifert surfaces of K are framed cobordant relative boundary to each other. Hence the values Sign V and c(V) are independent of the choice of V. We set

$$\sigma(S^{n+2}, K) = \begin{cases} \operatorname{Sign} V & \text{if} & n+1 \equiv 0 \text{ (4),} \\ c(V) & \text{if} & n+1 = 2^k - 2 \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 4.2. Suppose  $\langle m \rangle = 1$  for  $(M^{n+2}, L^n)$  and n is odd  $\geqslant 5$ . Then  $(S^{n+2}, K) \in I_0(M, L)$  if  $\sigma(S^{n+2}, K) = 0$ . In particular,  $I_0(M, L) = \mathcal{K}_n$  if neither  $n+1 \equiv 0$  (4) nor  $n+1=2^k-2$  for some k.

Combining this with Theorem 1.1, we obtain

COROLLARY 4.3. Suppose  $\langle m \rangle = 1$  for  $(M^{n+2}, L^n)$  and  $n+1 \equiv 0$  (4)  $(n \neq 3)$ . Then  $(S^{n+2}, K) \in I_0(M, L)$  if and only if  $\sigma(S^{n+2}, K) = 0$ .

The rest of this section is devoted to the proof of Proposition 4.2. Let K be an n-knot in  $S^{n+2}$  such that  $\sigma(S^{n+2}, K) = 0$ . We shall construct an s-cobordism relative boundary between  $E(L \ K)$  and E(L). The argument is rather more complicated than that of Proposition 3.6. We need some knowledge of surgery theory.

Step 1. Let  $V^{n+1}$  be a Seifert surface of K. Push the interior of V into the interior of  $D^{n+3}$  to make it transverse to the boundary  $S^{n+2}$  of  $D^{n+3}$ . We may assume that V is (n-1)/2-connected, if necessary, by doing framed surgery of V within  $D^{n+3}$ . In fact, this is the method used to prove that any n-knot is concordant to a simple knot (see [KW, Chap. IV]).

In the attempt to make V(n+1)/2-connected (and hence V is contractible by the Poincaré duality) by framed surgery of V within  $D^{n+3}$ , one encounters an obstruction. Namely a bunch of embedded (n+1)/2-spheres in V does

not necessarily extend to embedded (n+3)/2-disks whose interior lies in  $D^{n+3} - V$ .

But if we do framed surgery of V at the outside of  $D^{n+3}$  without touching boundary, i.e. if we do surgery on framed embeddings

$$(S^{(n+1)/2} \times D^{(n+1)/2} \times D^2, S^{(n+1)/2} \times D^{(n+1)/2} \times \{0\}) \to (D^{n+3}, V),$$

then we can make V(n+1)/2-connected because the obstruction is exactly  $\sigma(S^{n+2}, K)$  and it vanishes by the assumption. The ambient space is, however, not  $D^{n+3}$  any more. We denote by (W, D) the resulting framed oriented pair, where D is diffeomorphic to  $D^{n+1}$ .

Step 2. We construct a boundary preserving map h:

$$(W; N(D), E(D)) \rightarrow (D^{n+3}; N(D^{n+1}), E(D^{n+1}))$$

such that

(4.4) 
$$h|_{\partial W}: \partial W = S^{n+2} \to \partial D^{n+3} = S^{n+2}$$
 is a homotopy equivalence,

(4.5) 
$$h|_{N(D)}: N(D) \to N(D^{n+1})$$
 is a diffeomorphism,

where N denotes a closed tubular neighborhood and  $D^{n+1} \subset D^{n+3}$  is standardly embedded.

Since D is diffeomorphic to  $D^{n+1}$ , there is a diffeomorphism

$$g: (D^{n+1} \times D^2, D^{n+1} \times \{0\}) \to (N(D), D).$$

Here  $D^{n+1} \times D^2$  can be naturally identified with  $N(D^{n+1})$ ; so we define

$$(4.6) h|_{N(D)} = g^{-1}$$

First we extend  $h|_{\partial W \cap \partial N(D)} = h|_{\partial E(K)}$  to a map from E(K) to  $E(\partial D^{n+1}) = E(S^n)$ . The obstruction lies in groups

$$H^{q+1}(E(K), \partial E(K); \pi_a(E(S^n)))$$
.

Since  $E(S^n)$  is homotopy equivalent to  $S^1$ , it suffices to prove

(4.7) 
$$H^{q+1}(E(K), \partial E(K); \mathbf{Z}) = 0$$
 for  $q = 0, 1$ .

On the other hand we have

$$H^{q+1}(E(K), \partial E(K); \mathbf{Z}) \simeq H^{q+1}(S^{n+2}, N(K); \mathbf{Z})$$
 (by excision)  
 $\simeq \tilde{H}^{q}(N(K); \mathbf{Z})$  (if  $q+1 < n+2$ )  
 $\simeq \tilde{H}^{q}(S^{n}; \mathbf{Z})$   
 $= 0$  (if  $q \neq n$ )

Hence (4.7) is satisfied as  $n \ge 5$ .

Consequently we can extend  $h|_{N(D)}$  to a map

$$h \mid_{N(D) \cup \partial W} : (N(D) \cup \partial W, \partial W) \to (N(D^{n+1}) \cup \partial D^{n+3}, \partial D^{n+3}).$$

The local degree of  $h|_{\partial W}: \partial W \to \partial D^{n+3}$  is one because  $h|_{\partial W \cap N(D)} = h|_{N(K)}: N(K) \to N(S^n)$  is a diffeomorphism by (4.6) and  $h(E(K)) \subset E(S^n)$  by the construction. Since  $\partial W$  and  $\partial D^{n+3}$  are both  $S^{n+2}$ ,  $h|_{\partial W}$  is a homotopy equivalence. Hence (4.4) is satisfied. Moreover (4.5) is also satisfied by (4.6). In the sequel it suffices to extend  $h|_{\partial E(D)}$  to a map from E(D) to  $E(D^{n+1})$ . This time the obstruction lies in groups

$$H^{q+1}(E(D), \partial E(D); \pi_q(E(D^{n+1})))$$
.

Since  $E(D^{n+1})$  is homotopy equivalent to  $S^1$ , it suffices to prove

(4.8) 
$$H^{q+1}(E(D), \partial E(D); \mathbf{Z}) = 0$$
 for  $q = 0, 1$ .

By excision we have

$$H^{q+1}(E(D), \partial E(D); \mathbf{Z}) \simeq H^{q+1}(W, N(D) \cup \partial W; \mathbf{Z}).$$

Remember that W is obtained from  $D^{n+3}$  by (n+1)/2-surgery. It implies that

$$\tilde{H}^{i}(W; \mathbf{Z}) = 0$$
 if  $i \neq (n+1)/2 + 1$ .

In particular

$$\tilde{H}^i(W; \mathbf{Z}) = 0$$
 for  $i \leq 3$ 

as  $n \ge 5$ . Therefore it follows from the exact sequence of the pair  $(W, N(D) \cup \partial W)$  that

$$H^{q+1}(W, N(D) \cup \partial W; \mathbf{Z}) \simeq \tilde{H}^{q}(N(D) \cup \partial W; \mathbf{Z})$$
 for  $q \leq 2$ .

Here the Mayer-Vietoris exact sequence of the triad  $(N(D) \cup \partial W; N(D), \partial W)$  shows that

$$\tilde{H}^q(N(D) \cup \partial W; \mathbf{Z}) = 0$$
 for  $q = 0, 1$ ,

because N(D) is contractible,  $\partial W = S^{n+2}$ , and  $N(D) \cap \partial W = S^n \times S^1$ . Hence (4.8) is satisfied, and we have obtained the desired map h.

Step 3. Since W is framed, the framing of the stable normal bundle  $\nu(W)$  of W induces a stable bundle map  $b:\nu(W)\to\nu(D^{n+3})$  which covers h. The triple  $\mathscr{B}=(W,h,b)$  is called a normal map.

The identity map  $Id: (M, L) \times I \rightarrow (M, L) \times I$  gives a normal map where the stable bundle map is also the identity. We shall denote the normal

map by  $\mathcal{B}_{Id} = ((M, L) \times I, Id, Id)$ . The maps h and Id are both diffeomorphisms on N(D) and  $N(L \times I)$  respectively; so one can do the boundary connected sum of  $\mathcal{B}$  and  $\mathcal{B}_{Id}$  at points of K and  $L \times \{1\}$ . This yields a new normal map  $\mathcal{B}_{Id} \nmid \mathcal{B} = (M \times I \nmid W, Id \nmid h, Id \nmid b)$ . Here we naturally identify the target space  $(M, L) \times I \nmid (D^{n+3}, D^{n+1})$  with  $(M, L) \times I$ . Since  $Id \nmid h$  is a diffeomorphism on  $N(L \times I \nmid D)$ , it gives a product structure on  $N(L \times I \nmid D)$ . Thus we get a cobordism  $E(L \times I \nmid D)$  relative boundary between  $E(L \mid K)$  and E(L).

Step 4.  $Id \nmid h|_{E(L)} : E(L) \to E(L) \times \{0\}$  (the 0-level) is the identity; so it is a simple homotopy equivalence. We shall observe that  $h_1 = Id \nmid h|_{E(L \not\parallel K)} : E(L \not\parallel K) \to E(L) \times \{1\}$  (the 1-level) is also a simple homotopy equivalence.

We have a decomposition

$$E(L \sharp K) = E(L) \cup E(K)$$

in the same sense as (3.7). Hence, similarly to (3.8) one can see

$$\pi_1(E(L \sharp K)) \simeq \pi_1(E(L))$$

where the inclusion map induces the isomorphism.

We can view  $E(L) \times \{1\}$  as  $E(L \sharp S^n)$  and we also have

$$E(L \sharp S^n) = E(L) \cup E(S^n).$$

Then the map  $h_1$  can be viewed as the identity on E(L) and h on E(K). This together with (4.9) shows that  $h_{1*}: \pi_1(E(L \# K)) \to \pi_1(E(L \# S^n))$  is an isomorphism.

As before we consider the map  $\tilde{h}_1: \tilde{E}(L \sharp K) \to \tilde{E}(L \sharp S^n)$  lifted to the universal covers. Since < m > = 1, we have a diagram

(4.10) 
$$\widetilde{E}(L \sharp K) = \widetilde{E}(L) \cup E(K) \times \Pi$$

$$\downarrow^{h_1} \downarrow \qquad \downarrow^{h_{|E(K)} \times Id}$$

$$\widetilde{E}(L \sharp S^n) = \widetilde{E}(L) \cup E(S^n) \times \Pi,$$

where  $\Pi = \pi_1(M-L)$  as before. Since  $h|_{E(K)}$  is a homology equivalence, the above diagram tells us that  $\tilde{h}_{1*}: H_q(\tilde{E}(L \,\sharp\, K); \mathbf{Z}) \to H_q(\tilde{E}(L \,\sharp\, S^n); \mathbf{Z})$  is an isomorphism as  $\mathbf{Z}[\Pi]$ -modules. Therefore  $h_1$  is a homotopy equivalence by the same reason as before.

The assumption < m > = 1 together with the above diagram tells us that  $\tau(h_1) \in Wh(\Pi)$  comes from an element of Wh(1). Hence  $\tau(h_1) = 0$  as Wh(1) = 0.

Step 5. By step 4  $\bar{h} = Id \nmid h \mid_{E(L \times I \nmid D)} : E(L \times I \nmid D) \to E(L \times I \nmid D^{n+1})$ =  $E(L \times I)$  is a simple homotopy equivalence on the boundary. We convert  $\bar{h}$  into a simple homotopy equivalence by surgery without touching the boundary. The obstruction  $\sigma(\bar{h})$  lies in an L-group  $L_{n+3}(\Pi, 1)$  where 1 denotes the trivial homomorphism from  $\Pi$  to  $\mathbb{Z}_2$  (note, since M is oriented and hence so is  $E(L \times I)$ , the orientation homomorphism:  $\Pi = \pi_1(E(L \times I)) \to \mathbb{Z}_2$  is trivial).

We have a diagram similar to (4.10):

$$E(L \times I \nmid D) = E(L \times I) \cup E(D)$$

$$\downarrow^{\bar{h}} \downarrow \qquad \qquad \downarrow^{\bar{h}}$$

$$E(L \times I \nmid D^{n+1}) = E(L \times I) \cup E(D^{n+1}).$$

The surgery obstruction  $\sigma(h)$  to converting h to a simple homotopy equivalence by surgery without touching the boundary lies in  $L_{n+3}(\mathbf{Z}, 1)$  because  $\pi_1(E(D^{n+1}))$  is isomorphic to  $\mathbf{Z}$ . The above diagram together with the assumption < m > = 1 tells us that

$$\sigma(\bar{h}) = \beta_* \alpha_* \sigma(h)$$

where  $\alpha_*: L_{n+3}(\mathbf{Z}, 1) \to L_{n+3}(1, 1)$  and  $\beta_*: L_{n+3}(1, 1) \to L_{n+3}(\Pi, 1)$  are the homomorphisms induced from the trivial homomorphisms  $\alpha: \mathbf{Z} \to 1$  and  $\beta: 1 \to \Pi$  respectively. It is well-known that

$$L_{n+3}(1, 1) \simeq \begin{cases} \mathbf{Z} & \text{if } n+3 \equiv 0 \text{ (4)}, \\ \mathbf{Z}_2 & \text{if } n+3 \equiv 2 \text{ (4)}. \end{cases}$$

As easily observed  $\alpha_*\sigma(h)$  is given by

$$\begin{cases} \text{Sign } W & \text{if} & n+3 \equiv 0 \text{ (4)} \\ c(W) & \text{if} & n+3 \equiv 2 \text{ (4)} \end{cases}$$

through the above isomorphism. Remember that W is framed cobordant to  $D^{n+3}$  relative boundary by the construction. Therefore those invariants vanish and hence  $\sigma(\bar{h}) = 0$ .

Consequently we have obtained a cobordism U' relative boundary between  $E(L \,\sharp\, K)$  and E(L) together with a simple homotopy equivalence  $F:U'\to E(L\times I)$  which is the identity on the 0-level. Let  $i_0\colon E(L)\to U'$  and  $j_0\colon E(L)\to E(L\times I)$  be the inclusion maps from the 0-level to the cobordisms. Since  $F\circ i_0=j_0\circ Id$  where  $Id\colon E(L)\to E(L)$  denotes the identity map, we have

$$\tau(F) + F_* \tau(i_0) = \tau(j_0) + j_{0*} \tau(Id)$$

(see [Ml, Lemma 7.8]). Here F,  $j_0$ , and Id are all simple homotopy equivalences; so these Whitehead torsions vanish. Hence it follows that  $\tau(i_0) = 0$ , because  $F_* : Wh(\pi_1(U')) \to Wh(\pi_1(E(L \times I)))$  is an isomorphism. This means that U' is an s-cobordism. Therefore  $(S^{n+2}, K) \in I_0(M, L)$  by Lemma 1.6. Q.E.D.

## § 5. Type 3 case

In this section we treat the case where < m > or [m] is of order p (p is not necessarily a prime number). We begin with

LEMMA 5.1. Suppose [m] is of order p. Then if  $(S^{n+2}, K) \in I(M, L)$ , then  $(S^{n+2}, K)_p$  is a homotopy (n+2)-sphere.

*Proof.* Let r be the order of Tor  $H_1(M-L; \mathbf{Z})$ , and let  $\gamma$  be the canonical epimorphism  $\pi_1(M-L) \to H_1(M-L; \mathbf{Z}) \otimes \mathbf{Z}_r$ . Since the order of  $\gamma(< m >)$  is p, we obtain the desired result by an argument similar to the proof of Lemma 2.1. Q.E.D.

If  $p \ge 2$ , there are infinitely many knots  $(S^{n+2}, K)$  such that  $(S^{n+2}, K)_p$  is not a homotopy (n+2)-sphere; so Lemma 5.1 shows that  $I(M, L) \subset \mathcal{K}_n$  for such (M, L).

The rest of this section is devoted to looking for a non-trivial knot in I(M, L) or  $I_0(M, L)$ . We will extend Proposition 3.6 and 4.2 to the case where < m > is of order p. Lemma 5.1 reminds us of counterexamples to the generalized Smith conjecture.

Let  $(S^{n+2}, K)$  be an *n*-knot which bounds a disk pair  $(D^{n+3}, D)$  such that  $(D^{n+3}, D)_p$  is a homotopy (n+3)-disk. Since  $(S^{n+2}, K)_p$  is the boundary of  $(D^{n+3}, D)_p$ ,  $(S^{n+2}, K)_p$  is a homotopy (n+2)-sphere. If  $n+3 \ge 5$ , then  $(D^{n+3}, D)_p$  is diffeomorphic to  $D^{n+3}$  and hence  $(S^{n+2}, K)_p$  is diffeomorphic to  $S^{n+2}$ .

The p-fold branched cyclic covering  $(D^{n+3}, D)_p$  supports a  $\mathbb{Z}_p$ -action with the branch set D as the fixed point set. Let  $E(D)_p$  be the exterior of D in  $(D^{n+3}, D)_p$  and let  $\rho: S^1 \to E(D)_p$  be an equivariant embedding of a meridian of D in  $E(D)_p$ , where the standard free  $\mathbb{Z}_p$ -action is considered on  $S^1$ . Since  $\rho$  is a homology equivalence and equivariant, the Whitehead torsion of  $\rho$  is defined in  $Wh(\mathbb{Z}_p)$ . Clearly it is independent of the choice of  $\rho$ ; so we shall denote it by  $\tau_p(D^{n+3}, D)$ .

The following theorem is an extension of Proposition 3.6.