| Zeitschrift: | L'Enseignement Mathématique                              |
|--------------|--|
| Herausgeber: | Commission Internationale de l'Enseignement Mathématique |
| Band:        | 35 (1989)  |
| Heft:        | 1-2: L'ENSEIGNEMENT MATHÉMATIQUE                         |
|              |  |
| Artikel:     | KNOTTING CODIMENSION 2 SUBMANIFOLDS LOCALLY              |
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| Kapitel:     | §3. Type 2 case  |
| DOI:         | https://doi.org/10.5169/seals-57360                      |

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# § 3. Type 2 case

In this section and the next section, we treat the case where a meridian of  $L^n$  in  $M^{n+2}$  is null homotopic in M - L. The following lemma follows from [Li, Lemma 1]. We shall give an alternative proof which is interesting by itself (the argument is also given in [Ms, Theorem 4.2]).

LEMMA 3.1.  $I(S^n \times S^2, S^n \times \{*\}) = \mathscr{K}_n \text{ if } n \ge 3.$ 

*Proof.* Let  $(S^{n+2}, K)$  be an *n*-knot and consider  $(S^n \times S^2, S^n \times \{*\})$   $\# (S^{n+2}, K)$ . A subset  $S^n \times \{*\}$   $K \cup \{x_0\} \times S^2$   $(x_0 \in S^n)$  is exactly the wedge sum of  $S^n$  and  $S^2$ . As easily observed the complement of an open regular neighborhood of the subset is contractible and hence diffeomorphic to  $D^{n+2}$  as  $n + 2 \ge 5$ . This means that one can express

$$(S^n \times S^2, S^n \times \{*\}) \ \# (S^{n+2}, K) = (S^n \times S^2, S^n \times \{*\}) \ \# \Sigma$$

where  $\Sigma$  is a homotopy (n+2)-sphere and the connected sum at the right hand side is done away from the submanifold  $S^n \times \{*\}$ .

On the other hand the ambient manifold must be diffeomorphic to  $S^n \times S^2$  because it is the connected sum of  $S^n \times S^2$  with  $S^{n+2}$ . These mean that  $\Sigma$  belongs to the inertia group of  $S^n \times S^2$ . But the group is trivial ([Sc]), so  $\Sigma$  must be the standard sphere. This proves the lemma. Q.E.D.

We shall denote by  $\langle m \rangle$  the class in  $\pi_1(M-L)$  represented by a meridian of L in M.

LEMMA 3.2. Suppose M is spin, L is diffeomorphic to  $S^n$ , and  $n \ge 3$ . If  $\langle m \rangle = 1$  for (M, L), then  $(M, L) = (S^n \times S^2, S^n \times \{*\}) \# M'$  with a closed oriented manifold M' of dimension n + 2.

*Proof.* Since  $\langle m \rangle = 1$  and dim  $M \ge 5$ , the meridian *m* bounds a 2-disk in M - L. Therefore  $L \lor S^2$  is embedded in *M*. The normal bundle to *L* in *M* is trivial, because it is classified by the Euler class sitting in  $H^2(L; \mathbb{Z})$  and  $H^2(L; \mathbb{Z}) = 0$  as  $L = S^n$  and  $n \ge 3$ . The normal bundle of the embedded  $S^2$  is also trivial, because it is classified by the second Stiefel-Whitney class and it vanishes as *M* is spin. Hence the closed regular neighborhood of  $L \lor S^2$  in *M* is diffeomorphic to that of  $S^n \lor S^2$  naturally embedded in  $S^n \times S^2$ . In particular its boundary is diffeomorphic to  $S^{n+1}$ . This implies the lemma. Q.E.D.

Remark 3.3. A similar argument works even if M is not spin. But this time two cases arise according as the normal bundle of the embedded  $S^2$  is trivial or not. If it is trivial, then the same conclusion as above holds. If it is not trivial, we have

$$(M, L) = (S^n \tilde{\times} S^2, S^n) \# M'$$

Here  $S^n \times S^2$  denotes the total space of the sphere bundle associated with the nontrivial (n+1)-dimensional vector bundle over  $S^2$  (note that it is unique as  $\pi_1(SO(n+1)) \simeq Z_2$  for  $n \ge 2$ ) and the submanifold  $S^n$  denotes a fiber.

Combining Lemma 3.1 with 3.2, we obtain

THEOREM 3.4. Suppose M is spin, L is diffeomorphic to  $S^n$ , and  $n \ge 3$ . Then if  $\langle m \rangle = 1$  for (M, L), then  $I(M, L) = \mathcal{K}_n$ .

*Remark* 3.5. If the inertia group  $I(S^n \times S^2)$  is trivial, then the same argument as the proof of Lemma 3.1 proves that  $I(S^n \times S^2, S^n) = \mathscr{K}_n$  and hence one could drop the spin condition for M by Remark 3.3.

If  $L \neq S^n$ , then the above argument does not work. For a general L we construct an s-cobordism between pairs  $(M, L) \notin (S^{n+2}, K)$  and (M, L) and apply lemma 1.6. We denote the set of all null-cobordant *n*-knots by  $\mathscr{K}_n^0$ . According to Kervaire [K] (cf. [KW, Chap. IV])  $\mathscr{K}_n = \mathscr{K}_n^0$  if *n* is even, but  $\mathscr{K}_n \neq \mathscr{K}_n^0$  if *n* is odd.

PROPOSITION 3.6. Suppose  $\langle m \rangle = 1$  for  $(M^{n+2}, L^n)$  and  $n \ge 3$ . Then  $I_0(M, L)$  contains  $\mathscr{K}_n^0$ . In particular, if n is even  $\ge 4$ , then  $I_0(M, L) = I(M, L) = \mathscr{K}_n$ .

*Proof.* Let  $(S^{n+2}, K)$  bound a disk pair  $(D^{n+3}, D)$ , where D is a (n+1)-disk. The boundary connected sum  $(M, L) \times I \nmid (D^{n+3}, D)$  at the 1-level gives a cobordism between (M, L) and  $(M, L) \notin (S^{n+2}, K)$ .

We shall check the conditions (1) and (2) in Lemma 1.6 for this cobordism. First, since D is diffeomorphic to  $D^{n+1}$ ,  $L \times I \nmid D$  is diffeomorphic to  $L \times I$ ; so (1) is satisfied. Hence  $E(L \times I \nmid D)$  gives a cobordism relative boundary between E(L) and  $E(L \not K)$ . We note that

$$(3.7) E(L \times I \natural D) = E(L \times I) \cup E(D)$$

where  $E(L \times I)$  and E(D) are pasted together along  $D^{n+1} \times S^1$  embedded in their boundaries. The  $S^1$  factor corresponds to meridians of  $L \times I$  and D. Then the van Kampen's theorem says that

$$\pi_1(E(L \times I \nmid D)) \simeq \pi_1(E(L \times I)) \underset{}{*} \pi_1(E(D))$$
$$\simeq \pi_1(E(L \times I)) * (\pi_1(E(D))/)$$

where the latter isomorphism is because  $\langle m \rangle = 1$  in  $\pi_1(E(L \times I))$  by the assumption. Since  $\pi_1(E(D))/\langle m \rangle \simeq \pi_1(D^{n+3}) \simeq \{1\}$ , we have

(3.8) 
$$\pi_1(E(L \times I \natural D)) \simeq \pi_1(E(L \times I)) \simeq \pi_1(E(L)) .$$

Here the inclusion map  $i: E(L) = E(L) \times \{0\} \rightarrow E(L \times I \nmid D)$  induces the isomorphism.

We shall observe that *i* is a simple homotopy equivalence. For that purpose we consider the lifting of *i* to the universal covers. Since the map  $\pi_1(E(D)) \rightarrow \pi_1(E(L \times I \nmid D))$  induced by the inclusion map is trivial as observed above, it follows from (3.7) that

(3.9) 
$$\tilde{E}(L \times I \nmid D) = \tilde{E}(L \times I) \cup E(D) \times \Pi$$

where  $\Pi = \pi_1(E(L \times I \nmid D)) = \pi_1(M-L)$  and  $\tilde{E}(L \times I)$  and  $E(D) \times \Pi$  are pasted together  $\Pi$ -equivariantly along  $D^{n+1} \times S^1 \times \Pi$  embedded in their boundaries. This means that  $\tilde{i}_*: H_q(\tilde{E}(L); \mathbb{Z}) \to H_q(\tilde{E}(L \times I \nmid D); \mathbb{Z})$  is an isomorphism as  $\mathbb{Z}[\Pi]$ -modules. Hence  $i_*: \pi_q(E(L)) \to \pi_q(E(L \times I \nmid D))$  is an isomorphism by Namioka's theorem (see [W11, §4]) and hence *i* is a homotopy equivalence.

The assumption  $\langle m \rangle = 1$  together with (3.9) tells us that the Whitehead torsion  $\tau(i) \in Wh(\Pi)$  of the map *i* comes from an element of Wh(1) through the map:  $Wh(1) \to Wh(\Pi)$  induced from the inclusion  $1 \to \Pi$ . However Wh(1) = 0 and hence  $\tau(i) = 0$ . This shows that  $E(L \times I \nmid D)$  is an s-cobordism relative boundary. The proposition then follows from Lemma 1.6. Q.E.D.

Proposition 3.6 gives a complete answer to the case where n is even  $\ge 4$ . It would be interesting to ask if the same conclusion still holds in the case n = 2.

In the next section we will improve Proposition 3.6 when n is odd  $\ge 5$ .

## § 4. AN IMPROVEMENT

Throughout this section we assume *n* is odd  $\geq 5$ . Let  $V^{n+1}$  be a Seifert surface of an *n*-knot *K* in  $S^{n+2}$ . The normal bundle to *V* in  $S^{n+2}$  is trivial. We give the stable normal bundle of  $S^{n+2}$  a canonical framing so that *V* can be viewed as a framed manifold.