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§ 3. Type 2 case

In this section and the next section, we treat the case where a meridian of L^n in M^{n+2} is null homotopic in $M - L$. The following lemma follows from [Li, Lemma 1]. We shall give an alternative proof which is interesting by itself (the argument is also given in [Ms, Theorem 4.2]).

Lemma 3.1. *$I(S^n \times S^2, S^n \times \{*\}) = \mathscr{K}_n$ if $n \geqslant 3$.*

Proof. Let (S^{n+2}, K) be an n-knot and consider $(S^n \times S^2, S^n \times \{*\}) \# (S^{n+2}, K)$. A subset $S^n \times \{*\}$ $K \cup \{x_0\} \times S^2$ $(x_0 \in S^n)$ is exactly the wedge sum of S^n and S^2. As easily observed the complement of an open regular neighborhood of the subset is contractible and hence diffeomorphic to D^{n+2} as $n + 2 \geqslant 5$. This means that one can express

$$(S^n \times S^2, S^n \times \{*\}) \# (S^{n+2}, K) = (S^n \times S^2, S^n \times \{*\}) \# \Sigma$$

where Σ is a homotopy $(n+2)$-sphere and the connected sum at the right hand side is done away from the submanifold $S^n \times \{*\}$.

On the other hand the ambient manifold must be diffeomorphic to $S^n \times S^2$ because it is the connected sum of $S^n \times S^2$ with S^{n+2}. These mean that Σ belongs to the inertia group of $S^n \times S^2$. But the group is trivial ([Sc]), so Σ must be the standard sphere. This proves the lemma. Q.E.D.

We shall denote by $<m>$ the class in $\pi_1(M-L)$ represented by a meridian of L in M.

Lemma 3.2. *Suppose M is spin, L is diffeomorphic to S^n, and $n \geqslant 3$. If $<m> = 1$ for (M, L), then $(M, L) = (S^n \times S^2, S^n \times \{*\}) \# M'$ with a closed oriented manifold M' of dimension $n + 2$.*

Proof. Since $<m> = 1$ and $\dim M \geqslant 5$, the meridian m bounds a 2-disk in $M - L$. Therefore $L \vee S^2$ is embedded in M. The normal bundle to L in M is trivial, because it is classified by the Euler class sitting in $H^2(L; \mathbf{Z})$ and $H^2(L; \mathbf{Z}) = 0$ as $L = S^n$ and $n \geqslant 3$. The normal bundle of the embedded S^2 is also trivial, because it is classified by the second Stiefel-Whitney class and it vanishes as M is spin. Hence the closed regular neighborhood of $L \vee S^2$ in M is diffeomorphic to that of $S^n \vee S^2$ naturally embedded in $S^n \times S^2$. In particular its boundary is diffeomorphic to S^{n+1}. This implies the lemma. Q.E.D.

Remark 3.3. A similar argument works even if M is not spin. But this time two cases arise according as the normal bundle of the embedded S^2 is trivial or not. If it is trivial, then the same conclusion as above holds. If it is not trivial, we have

$$(M, L) = (S^n \tilde{\times} S^2, S^n) \# M' .$$

Here $S^n \tilde{\times} S^2$ denotes the total space of the sphere bundle associated with the nontrivial $(n+1)$-dimensional vector bundle over S^2 (note that it is unique as $\pi_1(SO(n+1)) \simeq Z_2$ for $n \geqslant 2$) and the submanifold S^n denotes a fiber.

Combining Lemma 3.1 with 3.2, we obtain

THEOREM 3.4. *Suppose* M *is spin,* L *is diffeomorphic to* S^n, *and* $n \geqslant 3$. *Then if* $<m> = 1$ *for* (M, L), *then* $I(M, L) = \mathscr{K}_n$.

Remark 3.5. If the inertia group $I(S^n \tilde{\times} S^2)$ is trivial, then the same argument as the proof of Lemma 3.1 proves that $I(S^n \tilde{\times} S^2, S^n) = \mathscr{K}_n$ and hence one could drop the spin condition for M by Remark 3.3.

If $L \neq S^n$, then the above argument does not work. For a general L we construct an s-cobordism between pairs $(M, L) \# (S^{n+2}, K)$ and (M, L) and apply lemma 1.6. We denote the set of all null-cobordant n-knots by $\mathscr{K}_n^0$. According to Kervaire [K] (cf. [KW, Chap. IV]) $\mathscr{K}_n = \mathscr{K}_n^0$ if n is even, but $\mathscr{K}_n \neq \mathscr{K}_n^0$ if n is odd.

PROPOSITION 3.6. *Suppose* $<m> = 1$ *for* (M^{n+2}, L^n) *and* $n \geqslant 3$. *Then* $I_0(M, L)$ *contains* $\mathscr{K}_n^0$. *In particular, if* n *is even* $\geqslant 4$, *then* $I_0(M, L) = I(M, L) = \mathscr{K}_n$.

Proof. Let (S^{n+2}, K) bound a disk pair (D^{n+3}, D), where D is a $(n+1)$-disk. The boundary connected sum $(M, L) \times I \natural (D^{n+3}, D)$ at the 1-level gives a cobordism between (M, L) and $(M, L) \# (S^{n+2}, K)$.

We shall check the conditions (1) and (2) in Lemma 1.6 for this cobordism. First, since D is diffeomorphic to D^{n+1}, $L \times I \natural D$ is diffeomorphic to $L \times I$; so (1) is satisfied. Hence $E(L \times I \natural D)$ gives a cobordism relative boundary between $E(L)$ and $E(L \# K)$. We note that

$$E(L \times I \natural D) = E(L \times I) \cup E(D) \tag{3.7}$$

where $E(L \times I)$ and $E(D)$ are pasted together along $D^{n+1} \times S^1$ embedded in their boundaries. The S^1 factor corresponds to meridians of $L \times I$ and D. Then the van Kampen's theorem says that

$$\pi_1(E(L\times I \natural D)) \simeq \pi_1(E(L\times I)) \underset{<m>}{*} \pi_1(E(D))$$
$$\simeq \pi_1(E(L\times I)) * (\pi_1(E(D))/<m>)$$

where the latter isomorphism is because $<m> = 1$ in $\pi_1(E(L\times I))$ by the assumption. Since $\pi_1(E(D))/<m> \simeq \pi_1(D^{n+3}) \simeq \{1\}$, we have

$$\pi_1(E(L\times I \natural D)) \simeq \pi_1(E(L\times I)) \simeq \pi_1(E(L)). \tag{3.8}$$

Here the inclusion map $i\colon E(L) = E(L) \times \{0\} \to E(L\times I \natural D)$ induces the isomorphism.

We shall observe that i is a simple homotopy equivalence. For that purpose we consider the lifting of i to the universal covers. Since the map $\pi_1(E(D)) \to \pi_1(E(L\times I \natural D))$ induced by the inclusion map is trivial as observed above, it follows from (3.7) that

$$\tilde{E}(L\times I \natural D) = \tilde{E}(L\times I) \cup E(D) \times \Pi \tag{3.9}$$

where $\Pi = \pi_1(E(L\times I \natural D)) = \pi_1(M-L)$ and $\tilde{E}(L\times I)$ and $E(D) \times \Pi$ are pasted together Π-equivariantly along $D^{n+1} \times S^1 \times \Pi$ embedded in their boundaries. This means that $\tilde{i}_*\colon H_q(\tilde{E}(L);\mathbf{Z}) \to H_q(\tilde{E}(L\times I \natural D);\mathbf{Z})$ is an isomorphism as $\mathbf{Z}[\Pi]$-modules. Hence $i_*\colon \pi_q(E(L)) \to \pi_q(E(L\times I \natural D))$ is an isomorphism by Namioka's theorem (see [Wl1, § 4]) and hence i is a homotopy equivalence.

The assumption $<m> = 1$ together with (3.9) tells us that the Whitehead torsion $\tau(i) \in Wh(\Pi)$ of the map i comes from an element of $Wh(1)$ through the map: $Wh(1) \to Wh(\Pi)$ induced from the inclusion $1 \to \Pi$. However $Wh(1) = 0$ and hence $\tau(i) = 0$. This shows that $E(L\times I \natural D)$ is an s-cobordism relative boundary. The proposition then follows from Lemma 1.6. Q.E.D.

Proposition 3.6 gives a complete answer to the case where n is even $\geqslant 4$. It would be interesting to ask if the same conclusion still holds in the case $n = 2$.

In the next section we will improve Proposition 3.6 when n is odd $\geqslant 5$.

§ 4. An improvement

Throughout this section we assume n is odd $\geqslant 5$. Let V^{n+1} be a Seifert surface of an n-knot K in S^{n+2}. The normal bundle to V in S^{n+2} is trivial. We give the stable normal bundle of S^{n+2} a canonical framing so that V can be viewed as a framed manifold.