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## § 3. TYPE 2 CASE

In this section and the next section, we treat the case where a meridian of  $L^n$  in  $M^{n+2}$  is null homotopic in  $M - L$ . The following lemma follows from [Li, Lemma 1]. We shall give an alternative proof which is interesting by itself (the argument is also given in [Ms, Theorem 4.2]).

LEMMA 3.1.  $I(S^n \times S^2, S^n \times \{*\}) = \mathcal{K}_n$  if  $n \geq 3$ .

*Proof.* Let  $(S^{n+2}, K)$  be an  $n$ -knot and consider  $(S^n \times S^2, S^n \times \{*\}) \# (S^{n+2}, K)$ . A subset  $S^n \times \{*\} \cup K \cup \{x_0\} \times S^2$  ( $x_0 \in S^n$ ) is exactly the wedge sum of  $S^n$  and  $S^2$ . As easily observed the complement of an open regular neighborhood of the subset is contractible and hence diffeomorphic to  $D^{n+2}$  as  $n + 2 \geq 5$ . This means that one can express

$$(S^n \times S^2, S^n \times \{*\}) \# (S^{n+2}, K) = (S^n \times S^2, S^n \times \{*\}) \# \Sigma$$

where  $\Sigma$  is a homotopy  $(n+2)$ -sphere and the connected sum at the right hand side is done away from the submanifold  $S^n \times \{*\}$ .

On the other hand the ambient manifold must be diffeomorphic to  $S^n \times S^2$  because it is the connected sum of  $S^n \times S^2$  with  $S^{n+2}$ . These mean that  $\Sigma$  belongs to the inertia group of  $S^n \times S^2$ . But the group is trivial ([Sc]), so  $\Sigma$  must be the standard sphere. This proves the lemma. Q.E.D.

We shall denote by  $\langle m \rangle$  the class in  $\pi_1(M - L)$  represented by a meridian of  $L$  in  $M$ .

LEMMA 3.2. Suppose  $M$  is spin,  $L$  is diffeomorphic to  $S^n$ , and  $n \geq 3$ . If  $\langle m \rangle = 1$  for  $(M, L)$ , then  $(M, L) = (S^n \times S^2, S^n \times \{*\}) \# M'$  with a closed oriented manifold  $M'$  of dimension  $n + 2$ .

*Proof.* Since  $\langle m \rangle = 1$  and  $\dim M \geq 5$ , the meridian  $m$  bounds a 2-disk in  $M - L$ . Therefore  $L \vee S^2$  is embedded in  $M$ . The normal bundle to  $L$  in  $M$  is trivial, because it is classified by the Euler class sitting in  $H^2(L; \mathbf{Z})$  and  $H^2(L; \mathbf{Z}) = 0$  as  $L = S^n$  and  $n \geq 3$ . The normal bundle of the embedded  $S^2$  is also trivial, because it is classified by the second Stiefel-Whitney class and it vanishes as  $M$  is spin. Hence the closed regular neighborhood of  $L \vee S^2$  in  $M$  is diffeomorphic to that of  $S^n \vee S^2$  naturally embedded in  $S^n \times S^2$ . In particular its boundary is diffeomorphic to  $S^{n+1}$ . This implies the lemma. Q.E.D.

*Remark 3.3.* A similar argument works even if  $M$  is not spin. But this time two cases arise according as the normal bundle of the embedded  $S^2$  is trivial or not. If it is trivial, then the same conclusion as above holds. If it is not trivial, we have

$$(M, L) = (S^n \tilde{\times} S^2, S^n) \# M'.$$

Here  $S^n \tilde{\times} S^2$  denotes the total space of the sphere bundle associated with the nontrivial  $(n+1)$ -dimensional vector bundle over  $S^2$  (note that it is unique as  $\pi_1(SO(n+1)) \simeq \mathbb{Z}_2$  for  $n \geq 2$ ) and the submanifold  $S^n$  denotes a fiber.

Combining Lemma 3.1 with 3.2, we obtain

**THEOREM 3.4.** *Suppose  $M$  is spin,  $L$  is diffeomorphic to  $S^n$ , and  $n \geq 3$ . Then if  $\langle m \rangle = 1$  for  $(M, L)$ , then  $I(M, L) = \mathcal{K}_n$ .*

*Remark 3.5.* If the inertia group  $I(S^n \tilde{\times} S^2)$  is trivial, then the same argument as the proof of Lemma 3.1 proves that  $I(S^n \tilde{\times} S^2, S^n) = \mathcal{K}_n$  and hence one could drop the spin condition for  $M$  by Remark 3.3.

If  $L \neq S^n$ , then the above argument does not work. For a general  $L$  we construct an  $s$ -cobordism between pairs  $(M, L) \# (S^{n+2}, K)$  and  $(M, L)$  and apply lemma 1.6. We denote the set of all null-cobordant  $n$ -knots by  $\mathcal{K}_n^0$ . According to Kervaire [K] (cf. [KW, Chap. IV])  $\mathcal{K}_n = \mathcal{K}_n^0$  if  $n$  is even, but  $\mathcal{K}_n \neq \mathcal{K}_n^0$  if  $n$  is odd.

**PROPOSITION 3.6.** *Suppose  $\langle m \rangle = 1$  for  $(M^{n+2}, L^n)$  and  $n \geq 3$ . Then  $I_0(M, L)$  contains  $\mathcal{K}_n^0$ . In particular, if  $n$  is even  $\geq 4$ , then  $I_0(M, L) = I(M, L) = \mathcal{K}_n$ .*

*Proof.* Let  $(S^{n+2}, K)$  bound a disk pair  $(D^{n+3}, D)$ , where  $D$  is a  $(n+1)$ -disk. The boundary connected sum  $(M, L) \times I \natural (D^{n+3}, D)$  at the 1-level gives a cobordism between  $(M, L)$  and  $(M, L) \# (S^{n+2}, K)$ .

We shall check the conditions (1) and (2) in Lemma 1.6 for this cobordism. First, since  $D$  is diffeomorphic to  $D^{n+1}$ ,  $L \times I \natural D$  is diffeomorphic to  $L \times I$ ; so (1) is satisfied. Hence  $E(L \times I \natural D)$  gives a cobordism relative boundary between  $E(L)$  and  $E(L \# K)$ . We note that

$$(3.7) \quad E(L \times I \natural D) = E(L \times I) \cup E(D)$$

where  $E(L \times I)$  and  $E(D)$  are pasted together along  $D^{n+1} \times S^1$  embedded in their boundaries. The  $S^1$  factor corresponds to meridians of  $L \times I$  and  $D$ . Then the van Kampen's theorem says that

$$\begin{aligned}\pi_1(E(L \times I \pitchfork D)) &\simeq \pi_1(E(L \times I)) \underset{<m>}{\ast} \pi_1(E(D)) \\ &\simeq \pi_1(E(L \times I)) \ast (\pi_1(E(D)) / <m>)\end{aligned}$$

where the latter isomorphism is because  $<m> = 1$  in  $\pi_1(E(L \times I))$  by the assumption. Since  $\pi_1(E(D)) / <m> \simeq \pi_1(D^{n+3}) \simeq \{1\}$ , we have

$$(3.8) \quad \pi_1(E(L \times I \pitchfork D)) \simeq \pi_1(E(L \times I)) \simeq \pi_1(E(L)).$$

Here the inclusion map  $i: E(L) = E(L) \times \{0\} \rightarrow E(L \times I \pitchfork D)$  induces the isomorphism.

We shall observe that  $i$  is a simple homotopy equivalence. For that purpose we consider the lifting of  $i$  to the universal covers. Since the map  $\pi_1(E(D)) \rightarrow \pi_1(E(L \times I \pitchfork D))$  induced by the inclusion map is trivial as observed above, it follows from (3.7) that

$$(3.9) \quad \tilde{E}(L \times I \pitchfork D) = \tilde{E}(L \times I) \cup E(D) \times \Pi$$

where  $\Pi = \pi_1(E(L \times I \pitchfork D)) = \pi_1(M - L)$  and  $\tilde{E}(L \times I)$  and  $E(D) \times \Pi$  are pasted together  $\Pi$ -equivariantly along  $D^{n+1} \times S^1 \times \Pi$  embedded in their boundaries. This means that  $\tilde{i}_*: H_q(\tilde{E}(L); \mathbf{Z}) \rightarrow H_q(\tilde{E}(L \times I \pitchfork D); \mathbf{Z})$  is an isomorphism as  $\mathbf{Z}[\Pi]$ -modules. Hence  $i_*: \pi_q(E(L)) \rightarrow \pi_q(E(L \times I \pitchfork D))$  is an isomorphism by Namioka's theorem (see [W11, § 4]) and hence  $i$  is a homotopy equivalence.

The assumption  $<m> = 1$  together with (3.9) tells us that the Whitehead torsion  $\tau(i) \in Wh(\Pi)$  of the map  $i$  comes from an element of  $Wh(1)$  through the map:  $Wh(1) \rightarrow Wh(\Pi)$  induced from the inclusion  $1 \rightarrow \Pi$ . However  $Wh(1) = 0$  and hence  $\tau(i) = 0$ . This shows that  $E(L \times I \pitchfork D)$  is an  $s$ -cobordism relative boundary. The proposition then follows from Lemma 1.6. Q.E.D.

Proposition 3.6 gives a complete answer to the case where  $n$  is even  $\geq 4$ . It would be interesting to ask if the same conclusion still holds in the case  $n = 2$ .

In the next section we will improve Proposition 3.6 when  $n$  is odd  $\geq 5$ .

#### § 4. AN IMPROVEMENT

Throughout this section we assume  $n$  is odd  $\geq 5$ . Let  $V^{n+1}$  be a Seifert surface of an  $n$ -knot  $K$  in  $S^{n+2}$ . The normal bundle to  $V$  in  $S^{n+2}$  is trivial. We give the stable normal bundle of  $S^{n+2}$  a canonical framing so that  $V$  can be viewed as a framed manifold.