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## KNOTTING CODIMENSION 2 SUBMANIFOLDS LOCALLY

by Mikiya MASUDA and Makoto SAKUMA

## INTRODUCTION

Let  $L$  be a connected oriented  $n$ -dimensional closed manifold smoothly embedded in a connected oriented  $(n+2)$ -dimensional closed manifold  $M$ , and let  $K$  be an oriented  $n$ -dimensional smooth knot in the oriented  $S^{n+2}$ . Then we consider the connected sum  $(M, L) \# (S^{n+2}, K)$ . In other words, we knot  $L$  locally using  $K$ . It yields another embedding of  $L$  in  $M$ ; however, it does not always give a new embedding. In fact, the lightbulb theorem says that the connected sum of  $(S^2 \times S^1, \{*\} \times S^1)$  with any knot in  $S^3$  is always equivalent to the original embedding. Moreover, by the prime decomposition theorem for knots in 3-manifolds [My],  $(S^2 \times S^1, \{*\} \times S^1)$  is essentially the only embedding of a circle with the above property. Litherland [Li] has generalized the lightbulb theorem to the higher dimensional cases. In the appendix of [V], Viro exhibits an example of a 2-knot whose connected sum with the standard projective plane in  $S^4$  does not change the isotopy type of the projective plane. (See also [La].)

The purpose of this paper is to study under what conditions this phenomenon occurs (or does not occur). The first named author [Ms] studied this problem when the codimension is greater than 2.

Put it in another way. Let  $\mathcal{K}_n$  be the set of isotopy classes of oriented  $n$ -knots diffeomorphic to  $S^n$  in the oriented  $S^{n+2}$ . The set forms an abelian monoid under connected sum for pairs. Analogously to the inertia group of a manifold, we define

$$I(M, L) = \{(S^{n+2}, K) \in \mathcal{K}_n \mid (M, L) \# (S^{n+2}, K) = (M, L)\}$$

where  $=$  in the parenthesis indicates that there is an orientation preserving diffeomorphism of pairs. The set forms a submonoid of  $\mathcal{K}_n$  and describes the effect of knotting  $L$  locally. We are also concerned with the following intermediate submonoid

$$I_0(M, L) = \{(S^{n+2}, K) \in I(M, L) \mid (M, L) \# (S^{n+2}, K) \equiv (M, L)\}$$

where  $\equiv$  indicates that there is an orientation preserving diffeomorphism of pairs which is concordant to the identity map as a diffeomorphism of the ambient space  $M$ .

Our results suggest that  $I(M, L)$  and  $I_0(M, L)$  depend only on the order of a meridian of  $L$  in  $\pi_1(M - L)$  or  $H_1(M - L; \mathbf{Z})$ . Roughly speaking, according as the order is infinite, 1, or  $p$  ( $1 < p < \infty$ ), they can be distinguished by (at least) these three types:

Type 1  $I(M, L) = \{0N\}$ ,

Type 2  $I(M, L) = \mathcal{K}_n$ ,  $I_0(M, L) = \ker \sigma$ ,

Type 3  $\{0\} \subsetneq I(M, L) \subsetneq \mathcal{K}_n$ ,  $\{0\} \subsetneq I_0(M, L) \subsetneq \ker \sigma$ ,

(see section 4 for  $\sigma(S^{n+2}, K)$ ).

We refer the reader to 1.1, 2.6, 3.4, 5.1, 5.2, and 5.8 for the precise statement.

This paper consists of five sections. In Section 1, we deduce a necessary condition for  $I_0(M, L)$ , which is valid for any  $(M, L)$ . We treat type 1 in Section 2. Type 2 is discussed in Sections 3, 4 and type 3 is discussed in Section 5. We will find that type 3 is closely related to the generalized Smith conjecture.

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### § 1. GENERAL REMARKS ON $I_0(M, L)$

It is known (and it is easily verified) that the signature of a Seifert surface of an oriented  $n$ -knot  $K$  in  $S^{n+2}$  is independent of the choice of a Seifert surface; so it is an invariant of the oriented knot  $K$ . The invariant is called the signature of the knot  $K$  and denoted by  $\text{Sign}(S^{n+2}, K)$ . We note that  $\text{Sign}(S^{n+2}, K)$  is trivially zero unless  $n + 1 \equiv 0 \pmod{4}$ .

As is seen in Section 3, there is a pair  $(M^{n+2}, L^n)$  such that  $I(M, L) = \mathcal{K}_n$  for any  $n \geq 3$ . In contrast, we can deduce a necessary condition for  $I_0(M, L)$  which holds for any pair  $(M, L)$ .

**THEOREM 1.1.** *If  $(S^{n+2}, K) \in I_0(M, L)$ , then  $\text{Sign}(S^{n+2}, K) = 0$ .*

*Proof.* Let  $V$  be a Seifert surface of  $K$ . Since  $S^{n+2} = \partial D^{n+3}$ , we can push the interior of  $V$  into the interior of  $D^{n+3}$  so that  $V$  is transverse to  $S^{n+2}$ . This yields an oriented pair  $(D^{n+3}, V)$  having  $(S^{n+2}, K)$  as the boundary.

The boundary connected sum  $(M, L) \times I \# (D^{n+3}, V)$  gives a cobordism between  $(M, L) \# (S^{n+2}, K)$  and  $(M, L)$ . We note that the ambient space of the cobordism is diffeomorphic to  $M \times I$ . Since  $(S^{n+2}, K) \in I_0(M, L)$ , there is an orientation preserving diffeomorphism  $f: (M, L) \# (S^{n+2}, K) \rightarrow (M, L)$  which is concordant to the identity when regarded as a diffeomorphism of the ambient space  $M$ . We paste together  $(M, L) \# (S^{n+2}, K)$  and  $(M, L)$  by  $f$  to get an oriented pair of closed manifolds. Since  $f$  is concordant to the identity, the resulting ambient space is diffeomorphic to  $M \times S^1$ . We shall denote by  $X$  the resulting oriented closed submanifold of  $M \times S^1$ .

The additivity property of the signature (see [AS, p. 588]) says that

$$\text{Sign } X = \text{Sign } L \times I + \text{Sign } V = \text{Sign } V,$$

where  $\text{Sign } L \times I = 0$  follows easily from the definition of the signature of a manifold with boundary. By the Hirzebruch signature theorem (see [MS, § 19]) we have

$$\text{Sign } X = \mathcal{L}(X)[X]$$

where the right hand side means the Hirzebruch  $L$ -class  $\mathcal{L}(X)$  of  $X$  evaluated on the fundamental class  $[X]$  of  $X$ . In the sequel we shall show  $\mathcal{L}(X)[X] = 0$ .

Let  $j: X \rightarrow M \times S^1$  be the inclusion map. Then it is not difficult to see that

$$(1.2) \quad j_*[X] = [L \times S^1] \quad \text{in} \quad H_{n+1}(M \times S^1; \mathbf{Z})$$

where  $[L \times S^1]$  denotes the homology class represented by  $L \times S^1$ .

Let  $v$  be the normal bundle to  $X$  in  $M \times S^1$ . By the multiplicativity of  $L$ -class we have

$$(1.3) \quad \begin{aligned} \mathcal{L}(X) &= \mathcal{L}(v)^{-1} j^* \mathcal{L}(M \times S^1) \\ \mathcal{L}(M \times S^1) &= \mathcal{L}(M) \times \mathcal{L}(S^1) = \pi^* \mathcal{L}(M) \end{aligned}$$

where  $\pi: M \times S^1 \rightarrow M$  is the projection map. Since  $\dim v = 2$ , we have

$$(1.4) \quad \mathcal{L}(v) = 1 + p_1(v)/3 = 1 + e(v)^2/3$$

where  $p_1$  and  $e$  denote the first Pontrjagin class and the Euler class respectively.

On the other hand it is known that

$$(1.5) \quad e(v) = j^* j_!(1)$$

where  $j_! : H^q(X ; \mathbf{Z}) \rightarrow H^{q+2}(M \times S^1 ; \mathbf{Z})$  denotes the Gysin homomorphism and  $1 \in H^0(X ; \mathbf{Z})$  is the unit element. Remember the definition of  $j_!$ . It is defined so that the following diagram commutes:

$$\begin{array}{ccc} H^q(X ; \mathbf{Z}) & \xrightarrow{j_!} & H^{q+2}(M \times S^1 ; \mathbf{Z}) \\ \downarrow \cap[X] & & \downarrow \cap[M \times S^1] \\ H_{n+1-q}(X ; \mathbf{Z}) & \xrightarrow{j_*} & H_{n+1-q}(M \times S^1 ; \mathbf{Z}) \end{array}$$

where the vertical maps are the Poincaré dualities. It says that

$$j_!(1) \cap [M \times S^1] = j_*[X].$$

This together with (1.2) means that

$$j_!(1) \in \pi^* H^2(M ; \mathbf{Z}).$$

Hence it follows from (1.4) and (1.5) that

$$\mathcal{L}(v) \in j^* \pi^* H^*(M ; Q)$$

and hence

$$\mathcal{L}(X) \in j^* \pi^* H^*(M ; Q)$$

by (1.3). This together with (1.2) implies that

$$\mathcal{L}(X)[X] = 0. \quad \text{Q.E.D.}$$

Theorem 1.1 gives a necessary condition for  $(S^{n+2}, K)$  to belong to  $I_0(M, L)$ . When we consider the converse problem, i.e. the problem to find  $(S^{n+2}, K)$  in  $I_0(M, L)$ , we apply the relative  $s$ -cobordism theorem. We shall state it as a lemma for later convenience's sake.

**LEMMA 1.6.** *Suppose there exists a cobordism  $(U, Z)$  between  $(M, L)$   $\# (S^{n+2}, K)$  and  $(M, L)$  such that*

- (1)  $Z$  is diffeomorphic to  $L \times I$ ,
- (2) the exterior  $E(Z)$  of  $Z$  is an  $s$ -cobordism relative boundary.

*Then  $(S^{n+2}, K) \in I_0(M, L)$ .*

*Proof.* The relative  $s$ -cobordism theorem says that  $E(Z)$  is diffeomorphic to  $E(L) \times I$  where the diffeomorphism can be taken as the identity on  $E(L) \times \{0\}$  and  $(\partial E(L)) \times I$ . Therefore it extends to a diffeomorphism:  $(U, Z) \rightarrow (M, L) \times I$  which is the identity on the 0-level. This means that  $(S^{n+2}, K) \in I_0(M, L)$ . Q.E.D.

## § 2. TYPE 1 CASE

In this section we consider the case where a meridian of  $L^n$  in  $M^{n+2}$  has infinite order in  $H_1(M-L; \mathbf{Z})$ . We shall denote by  $[m]$  the homology class in  $H_1(M-L; \mathbf{Z})$  represented by a meridian  $m$  of  $L$  in  $M$ . For a manifold pair  $(X, Y)$  of codimension 2 and an epimorphism  $\gamma$  from  $\pi_1(X-Y)$  to a finite group, let  $(X, Y)_\gamma$  be the branched covering of  $(X, Y)$  corresponding to  $\gamma$ . Each knot group  $\pi_1(S^{n+2}-K)$  has a natural epimorphism to  $\mathbf{Z}_p$  for any positive integer  $p$ , and the corresponding  $p$ -fold branched cyclic covering of  $(S^{n+2}, K)$  is denoted by  $(S^{n+2}, K)_p$ .

LEMMA 2.1. *Suppose  $[m]$  is of infinite order. Then if  $(S^{n+2}, K) \in I(M, L)$  then  $(S^{n+2}, K)_p$  is a homotopy  $(n+2)$ -sphere for any positive integer  $p$ .*

*Proof.* Since  $[m]$  represents a nontrivial element in the finitely generated free abelian group  $B_1(M-L) \equiv H_1(M-L; \mathbf{Z})/\text{Tor } H_1(M-L; \mathbf{Z})$ , there is a positive integer  $r$  and a primitive element  $x$  in  $B_1(M-L)$  such that  $[m] = rx$  in  $B_1(M-L)$ . For each positive integer  $p$ , let  $\gamma_p$  be the canonical epimorphism  $\pi_1(M-L) \rightarrow B_1(M-L) \otimes \mathbf{Z}_{pr}$ . Noting the naturality of the homomorphism  $\gamma_p$ , we can see the following:

$$\begin{aligned} (M, L)_{\gamma_p} &= ((M, L) \# (S^{n+2}, K))_{\gamma_p \circ f_*} \\ &= (M, L)_{\gamma_p} \# d_p(S^{n+2}, K)_p \end{aligned}$$

Here  $f$  is a diffeomorphism  $(M, L) \# (S^{n+2}, K) \rightarrow (M, L)$  and  $d_p$  is the order of  $B_1(M-L) \otimes \mathbf{Z}_{pr}$  divided by  $p$ . Hence  $H_*(S^{n+2}, K)_p; \mathbf{Z} \simeq H_*(S^{n+2}; \mathbf{Z})$  and  $\pi_1(S^{n+2}, K)_p \simeq 1$  by the existence of prime decompositions of finitely generated groups into free products [Wg]. Q.E.D.

It is conjectured that those knots which satisfy the conclusion of the above lemma are trivial. In fact, for  $n = 1$ , it follows from the Smith conjecture [MB]. As a supporting evidence for higher dimensional cases, we have

LEMMA. *Suppose that  $(S^{n+2}, K)_p$  is a homotopy  $(n+2)$ -sphere for every positive integer  $p$ . Then the Alexander modules of  $K$  are trivial.*

*Proof.* Let  $\tilde{E}(K)$  be the infinite cyclic cover of the exterior  $E(K)$  of  $K$  in  $S^{n+2}$ , and let  $t$  denote the automorphism of the homology group of  $\tilde{E}(K)$  induced by the action of a meridian. Then, by the arguments of [Sm1],

we can see that  $t^p - 1: H_q(\tilde{E}(K); \mathbf{Z}_r) \rightarrow H_q(\tilde{E}(K); \mathbf{Z}_r)$  is an isomorphism for any positive integers  $p$ ,  $q$ , and  $r$ . Assume  $r$  is prime. Then  $H_q(\tilde{E}(K); \mathbf{Z}_r)$  is a finite abelian group, since it is a finitely generated torsion module over the principal ideal domain  $\mathbf{Z}_r\langle t \rangle$  (see [Le3, p. 8]). So the automorphism  $t$  on  $H_q(\tilde{E}(K); \mathbf{Z}_r)$  has a finite order, say  $d$ , and we have  $t^d - 1 = 0$ . Hence  $H_q(\tilde{E}(K); \mathbf{Z}_r) = 0$ , and by the universal coefficient theorem, the following holds for any prime  $r$  and any positive integer  $q$ :

$$(2.3) \quad H_q(\tilde{E}(K); \mathbf{Z}) \otimes \mathbf{Z}_r = 0$$

$$(2.4) \quad \text{Tor}(H_q(\tilde{E}(K); \mathbf{Z}), \mathbf{Z}_r) = 0$$

By (2.4),  $H_q(\tilde{E}(K); \mathbf{Z})$  has no nontrivial elements of finite order; so it has a square presentation matrix  $M(t)$  as a  $\mathbf{Z}\langle t \rangle$ -module by [Le3, Proposition 3.5]. By (2.3) the  $q$ -th Alexander polynomial  $\det M_q(t)$  ( $\in \mathbf{Z}\langle t \rangle$ ) is a unit mod.  $r$  for any prime  $r$ . Hence it is a unit in  $\mathbf{Z}\langle t \rangle$ , and we have  $H_q(\tilde{E}(K); \mathbf{Z}) = 0$  for any positive integer  $q$ . Q.E.D.

Thus, as a consequence of Lemmas 2.1 and 2.2 and the results of [Le2] and [T], we have the following:

**PROPOSITION 2.5.** *Suppose  $[m]$  is of infinite order. Then any knot in  $I(M, L)$  has trivial Alexander modules and is null cobordant.*

Hence the only obstruction for a knot  $(S^{n+2}, K)$  in  $I(M, L)$  to be trivial lies in the knot group  $\pi_1(S^{n+2} - K)$ . For the special case where  $[m]$  generates  $H_1(M - L)$ , we can apply the result of Maeda [Ma] (cf. [DF]), and obtain the following:

**THEOREM 2.6.** *Suppose  $n \geq 3$  and  $H_1(M - L)$  is the infinite cyclic group generated by  $[m]$ . Then  $I(M, L)$  is trivial.*

*Proof.* Let  $(S^{n+2}, K)$  be a knot in  $I(M, L)$ . Note that  $\pi_1(M - L)$  is isomorphic to the amalgamated free product  $\pi_1(M - L) *_{\langle m \rangle} \pi_1(S^{n+2} - K)$ .

Then we can conclude  $\pi_1(S^{n+2} - K) \simeq \mathbf{Z}$  by the result of [Ma] (cf. [DF]) which asserts the existence of a prime decomposition of a finitely presented group  $G$  with  $G/[G, G] \simeq \mathbf{Z}$  with respect to such amalgamated free products. Combined with Proposition 2.5, we see  $S^{n+2} - K$  is homotopy equivalent to a circle. Hence  $(S^{n+2}, K)$  is trivial by [Le1].

## § 3. TYPE 2 CASE

In this section and the next section, we treat the case where a meridian of  $L^n$  in  $M^{n+2}$  is null homotopic in  $M - L$ . The following lemma follows from [Li, Lemma 1]. We shall give an alternative proof which is interesting by itself (the argument is also given in [Ms, Theorem 4.2]).

LEMMA 3.1.  $I(S^n \times S^2, S^n \times \{*\}) = \mathcal{K}_n$  if  $n \geq 3$ .

*Proof.* Let  $(S^{n+2}, K)$  be an  $n$ -knot and consider  $(S^n \times S^2, S^n \times \{*\}) \# (S^{n+2}, K)$ . A subset  $S^n \times \{*\} \cup K \cup \{x_0\} \times S^2$  ( $x_0 \in S^n$ ) is exactly the wedge sum of  $S^n$  and  $S^2$ . As easily observed the complement of an open regular neighborhood of the subset is contractible and hence diffeomorphic to  $D^{n+2}$  as  $n + 2 \geq 5$ . This means that one can express

$$(S^n \times S^2, S^n \times \{*\}) \# (S^{n+2}, K) = (S^n \times S^2, S^n \times \{*\}) \# \Sigma$$

where  $\Sigma$  is a homotopy  $(n+2)$ -sphere and the connected sum at the right hand side is done away from the submanifold  $S^n \times \{*\}$ .

On the other hand the ambient manifold must be diffeomorphic to  $S^n \times S^2$  because it is the connected sum of  $S^n \times S^2$  with  $S^{n+2}$ . These mean that  $\Sigma$  belongs to the inertia group of  $S^n \times S^2$ . But the group is trivial ([Sc]), so  $\Sigma$  must be the standard sphere. This proves the lemma. Q.E.D.

We shall denote by  $\langle m \rangle$  the class in  $\pi_1(M - L)$  represented by a meridian of  $L$  in  $M$ .

LEMMA 3.2. Suppose  $M$  is spin,  $L$  is diffeomorphic to  $S^n$ , and  $n \geq 3$ . If  $\langle m \rangle = 1$  for  $(M, L)$ , then  $(M, L) = (S^n \times S^2, S^n \times \{*\}) \# M'$  with a closed oriented manifold  $M'$  of dimension  $n + 2$ .

*Proof.* Since  $\langle m \rangle = 1$  and  $\dim M \geq 5$ , the meridian  $m$  bounds a 2-disk in  $M - L$ . Therefore  $L \vee S^2$  is embedded in  $M$ . The normal bundle to  $L$  in  $M$  is trivial, because it is classified by the Euler class sitting in  $H^2(L; \mathbf{Z})$  and  $H^2(L; \mathbf{Z}) = 0$  as  $L = S^n$  and  $n \geq 3$ . The normal bundle of the embedded  $S^2$  is also trivial, because it is classified by the second Stiefel-Whitney class and it vanishes as  $M$  is spin. Hence the closed regular neighborhood of  $L \vee S^2$  in  $M$  is diffeomorphic to that of  $S^n \vee S^2$  naturally embedded in  $S^n \times S^2$ . In particular its boundary is diffeomorphic to  $S^{n+1}$ . This implies the lemma. Q.E.D.

*Remark 3.3.* A similar argument works even if  $M$  is not spin. But this time two cases arise according as the normal bundle of the embedded  $S^2$  is trivial or not. If it is trivial, then the same conclusion as above holds. If it is not trivial, we have

$$(M, L) = (S^n \tilde{\times} S^2, S^n) \# M'.$$

Here  $S^n \tilde{\times} S^2$  denotes the total space of the sphere bundle associated with the nontrivial  $(n+1)$ -dimensional vector bundle over  $S^2$  (note that it is unique as  $\pi_1(SO(n+1)) \simeq \mathbb{Z}_2$  for  $n \geq 2$ ) and the submanifold  $S^n$  denotes a fiber.

Combining Lemma 3.1 with 3.2, we obtain

**THEOREM 3.4.** *Suppose  $M$  is spin,  $L$  is diffeomorphic to  $S^n$ , and  $n \geq 3$ . Then if  $\langle m \rangle = 1$  for  $(M, L)$ , then  $I(M, L) = \mathcal{K}_n$ .*

*Remark 3.5.* If the inertia group  $I(S^n \tilde{\times} S^2)$  is trivial, then the same argument as the proof of Lemma 3.1 proves that  $I(S^n \tilde{\times} S^2, S^n) = \mathcal{K}_n$  and hence one could drop the spin condition for  $M$  by Remark 3.3.

If  $L \neq S^n$ , then the above argument does not work. For a general  $L$  we construct an  $s$ -cobordism between pairs  $(M, L) \# (S^{n+2}, K)$  and  $(M, L)$  and apply lemma 1.6. We denote the set of all null-cobordant  $n$ -knots by  $\mathcal{K}_n^0$ . According to Kervaire [K] (cf. [KW, Chap. IV])  $\mathcal{K}_n = \mathcal{K}_n^0$  if  $n$  is even, but  $\mathcal{K}_n \neq \mathcal{K}_n^0$  if  $n$  is odd.

**PROPOSITION 3.6.** *Suppose  $\langle m \rangle = 1$  for  $(M^{n+2}, L^n)$  and  $n \geq 3$ . Then  $I_0(M, L)$  contains  $\mathcal{K}_n^0$ . In particular, if  $n$  is even  $\geq 4$ , then  $I_0(M, L) = I(M, L) = \mathcal{K}_n$ .*

*Proof.* Let  $(S^{n+2}, K)$  bound a disk pair  $(D^{n+3}, D)$ , where  $D$  is a  $(n+1)$ -disk. The boundary connected sum  $(M, L) \times I \natural (D^{n+3}, D)$  at the 1-level gives a cobordism between  $(M, L)$  and  $(M, L) \# (S^{n+2}, K)$ .

We shall check the conditions (1) and (2) in Lemma 1.6 for this cobordism. First, since  $D$  is diffeomorphic to  $D^{n+1}$ ,  $L \times I \natural D$  is diffeomorphic to  $L \times I$ ; so (1) is satisfied. Hence  $E(L \times I \natural D)$  gives a cobordism relative boundary between  $E(L)$  and  $E(L \# K)$ . We note that

$$(3.7) \quad E(L \times I \natural D) = E(L \times I) \cup E(D)$$

where  $E(L \times I)$  and  $E(D)$  are pasted together along  $D^{n+1} \times S^1$  embedded in their boundaries. The  $S^1$  factor corresponds to meridians of  $L \times I$  and  $D$ . Then the van Kampen's theorem says that

$$\begin{aligned}\pi_1(E(L \times I \pitchfork D)) &\simeq \pi_1(E(L \times I)) \underset{<m>}{\ast} \pi_1(E(D)) \\ &\simeq \pi_1(E(L \times I)) \ast (\pi_1(E(D)) / <m>)\end{aligned}$$

where the latter isomorphism is because  $<m> = 1$  in  $\pi_1(E(L \times I))$  by the assumption. Since  $\pi_1(E(D)) / <m> \simeq \pi_1(D^{n+3}) \simeq \{1\}$ , we have

$$(3.8) \quad \pi_1(E(L \times I \pitchfork D)) \simeq \pi_1(E(L \times I)) \simeq \pi_1(E(L)).$$

Here the inclusion map  $i: E(L) = E(L) \times \{0\} \rightarrow E(L \times I \pitchfork D)$  induces the isomorphism.

We shall observe that  $i$  is a simple homotopy equivalence. For that purpose we consider the lifting of  $i$  to the universal covers. Since the map  $\pi_1(E(D)) \rightarrow \pi_1(E(L \times I \pitchfork D))$  induced by the inclusion map is trivial as observed above, it follows from (3.7) that

$$(3.9) \quad \tilde{E}(L \times I \pitchfork D) = \tilde{E}(L \times I) \cup E(D) \times \Pi$$

where  $\Pi = \pi_1(E(L \times I \pitchfork D)) = \pi_1(M - L)$  and  $\tilde{E}(L \times I)$  and  $E(D) \times \Pi$  are pasted together  $\Pi$ -equivariantly along  $D^{n+1} \times S^1 \times \Pi$  embedded in their boundaries. This means that  $\tilde{i}_*: H_q(\tilde{E}(L); \mathbf{Z}) \rightarrow H_q(\tilde{E}(L \times I \pitchfork D); \mathbf{Z})$  is an isomorphism as  $\mathbf{Z}[\Pi]$ -modules. Hence  $i_*: \pi_q(E(L)) \rightarrow \pi_q(E(L \times I \pitchfork D))$  is an isomorphism by Namioka's theorem (see [W11, § 4]) and hence  $i$  is a homotopy equivalence.

The assumption  $<m> = 1$  together with (3.9) tells us that the Whitehead torsion  $\tau(i) \in Wh(\Pi)$  of the map  $i$  comes from an element of  $Wh(1)$  through the map:  $Wh(1) \rightarrow Wh(\Pi)$  induced from the inclusion  $1 \rightarrow \Pi$ . However  $Wh(1) = 0$  and hence  $\tau(i) = 0$ . This shows that  $E(L \times I \pitchfork D)$  is an  $s$ -cobordism relative boundary. The proposition then follows from Lemma 1.6. Q.E.D.

Proposition 3.6 gives a complete answer to the case where  $n$  is even  $\geq 4$ . It would be interesting to ask if the same conclusion still holds in the case  $n = 2$ .

In the next section we will improve Proposition 3.6 when  $n$  is odd  $\geq 5$ .

#### § 4. AN IMPROVEMENT

Throughout this section we assume  $n$  is odd  $\geq 5$ . Let  $V^{n+1}$  be a Seifert surface of an  $n$ -knot  $K$  in  $S^{n+2}$ . The normal bundle to  $V$  in  $S^{n+2}$  is trivial. We give the stable normal bundle of  $S^{n+2}$  a canonical framing so that  $V$  can be viewed as a framed manifold.

Remember that  $\partial V = K = S^n$ . We make  $V$  contractible by framed surgery without touching the boundary. As is well known this is always possible in case  $\dim V = n + 1$  is odd. But in case  $n + 1$  is even, we encounter an obstruction which is detected by

$$\begin{cases} \text{Sign } V \in \mathbf{Z} & \text{if } n + 1 \equiv 0 \pmod{4} \\ c(V) \in \mathbf{Z}/2\mathbf{Z} & \text{if } n + 1 \equiv 2 \pmod{4} \end{cases}$$

where  $c(V)$  is the Kervaire invariant of  $V$ .

*Remark 4.1.* Since  $\partial V$  is diffeomorphic to  $S^n$ ,  $c(V) = 0$  if  $n + 1$  is not of the form  $2^k - 2$  ([Br]).

One can see that Seifert surfaces of  $K$  are framed cobordant relative boundary to each other. Hence the values  $\text{Sign } V$  and  $c(V)$  are independent of the choice of  $V$ . We set

$$\sigma(S^{n+2}, K) = \begin{cases} \text{Sign } V & \text{if } n + 1 \equiv 0 \pmod{4}, \\ c(V) & \text{if } n + 1 = 2^k - 2 \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

**PROPOSITION 4.2.** *Suppose  $\langle m \rangle = 1$  for  $(M^{n+2}, L^n)$  and  $n$  is odd  $\geq 5$ . Then  $(S^{n+2}, K) \in I_0(M, L)$  if  $\sigma(S^{n+2}, K) = 0$ . In particular,  $I_0(M, L) = \mathcal{K}_n$  if neither  $n + 1 \equiv 0 \pmod{4}$  nor  $n + 1 = 2^k - 2$  for some  $k$ .*

Combining this with Theorem 1.1, we obtain

**COROLLARY 4.3.** *Suppose  $\langle m \rangle = 1$  for  $(M^{n+2}, L^n)$  and  $n + 1 \equiv 0 \pmod{4}$  ( $n \neq 3$ ). Then  $(S^{n+2}, K) \in I_0(M, L)$  if and only if  $\sigma(S^{n+2}, K) = 0$ .*

The rest of this section is devoted to the proof of Proposition 4.2. Let  $K$  be an  $n$ -knot in  $S^{n+2}$  such that  $\sigma(S^{n+2}, K) = 0$ . We shall construct an  $s$ -cobordism relative boundary between  $E(L \# K)$  and  $E(L)$ . The argument is rather more complicated than that of Proposition 3.6. We need some knowledge of surgery theory.

*Step 1.* Let  $V^{n+1}$  be a Seifert surface of  $K$ . Push the interior of  $V$  into the interior of  $D^{n+3}$  to make it transverse to the boundary  $S^{n+2}$  of  $D^{n+3}$ . We may assume that  $V$  is  $(n-1)/2$ -connected, if necessary, by doing framed surgery of  $V$  within  $D^{n+3}$ . In fact, this is the method used to prove that any  $n$ -knot is concordant to a simple knot (see [KW, Chap. IV]).

In the attempt to make  $V$   $(n+1)/2$ -connected (and hence  $V$  is contractible by the Poincaré duality) by framed surgery of  $V$  within  $D^{n+3}$ , one encounters an obstruction. Namely a bunch of embedded  $(n+1)/2$ -spheres in  $V$  does

not necessarily extend to embedded  $(n+3)/2$ -disks whose interior lies in  $D^{n+3} - V$ .

But if we do framed surgery of  $V$  at the outside of  $D^{n+3}$  without touching boundary, i.e. if we do surgery on framed embeddings

$$(S^{(n+1)/2} \times D^{(n+1)/2} \times D^2, S^{(n+1)/2} \times D^{(n+1)/2} \times \{0\}) \rightarrow (D^{n+3}, V),$$

then we can make  $V$   $(n+1)/2$ -connected because the obstruction is exactly  $\sigma(S^{n+2}, K)$  and it vanishes by the assumption. The ambient space is, however, not  $D^{n+3}$  any more. We denote by  $(W, D)$  the resulting framed oriented pair, where  $D$  is diffeomorphic to  $D^{n+1}$ .

*Step 2.* We construct a boundary preserving map  $h$ :

$$(W; N(D), E(D)) \rightarrow (D^{n+3}; N(D^{n+1}), E(D^{n+1}))$$

such that

$$(4.4) \quad h|_{\partial W}: \partial W = S^{n+2} \rightarrow \partial D^{n+3} = S^{n+2} \quad \text{is a homotopy equivalence,}$$

$$(4.5) \quad h|_{N(D)}: N(D) \rightarrow N(D^{n+1}) \quad \text{is a diffeomorphism,}$$

where  $N$  denotes a closed tubular neighborhood and  $D^{n+1} \subset D^{n+3}$  is standardly embedded.

Since  $D$  is diffeomorphic to  $D^{n+1}$ , there is a diffeomorphism

$$g: (D^{n+1} \times D^2, D^{n+1} \times \{0\}) \rightarrow (N(D), D).$$

Here  $D^{n+1} \times D^2$  can be naturally identified with  $N(D^{n+1})$ ; so we define

$$(4.6) \quad h|_{N(D)} = g^{-1}$$

First we extend  $h|_{\partial W \cap \partial N(D)} = h|_{\partial E(K)}$  to a map from  $E(K)$  to  $E(\partial D^{n+1}) = E(S^n)$ . The obstruction lies in groups

$$H^{q+1}(E(K), \partial E(K); \pi_q(E(S^n))).$$

Since  $E(S^n)$  is homotopy equivalent to  $S^1$ , it suffices to prove

$$(4.7) \quad H^{q+1}(E(K), \partial E(K); \mathbf{Z}) = 0 \quad \text{for } q = 0, 1.$$

On the other hand we have

$$\begin{aligned} H^{q+1}(E(K), \partial E(K); \mathbf{Z}) &\simeq H^{q+1}(S^{n+2}, N(K); \mathbf{Z}) \quad (\text{by excision}) \\ &\simeq \tilde{H}^q(N(K); \mathbf{Z}) \quad (\text{if } q+1 < n+2) \\ &\simeq \tilde{H}^q(S^n; \mathbf{Z}) \\ &= 0 \quad (\text{if } q \neq n) \end{aligned}$$

Hence (4.7) is satisfied as  $n \geq 5$ .

Consequently we can extend  $h|_{N(D)}$  to a map

$$h|_{N(D) \cup \partial W}: (N(D) \cup \partial W, \partial W) \rightarrow (N(D^{n+1}) \cup \partial D^{n+3}, \partial D^{n+3}).$$

The local degree of  $h|_{\partial W}: \partial W \rightarrow \partial D^{n+3}$  is one because  $h|_{\partial W \cap N(D)} = h|_{N(K)}: N(K) \rightarrow N(S^n)$  is a diffeomorphism by (4.6) and  $h(E(K)) \subset E(S^n)$  by the construction. Since  $\partial W$  and  $\partial D^{n+3}$  are both  $S^{n+2}$ ,  $h|_{\partial W}$  is a homotopy equivalence. Hence (4.4) is satisfied. Moreover (4.5) is also satisfied by (4.6). In the sequel it suffices to extend  $h|_{\partial E(D)}$  to a map from  $E(D)$  to  $E(D^{n+1})$ . This time the obstruction lies in groups

$$H^{q+1}(E(D), \partial E(D); \pi_q(E(D^{n+1}))).$$

Since  $E(D^{n+1})$  is homotopy equivalent to  $S^1$ , it suffices to prove

$$(4.8) \quad H^{q+1}(E(D), \partial E(D); \mathbf{Z}) = 0 \quad \text{for } q = 0, 1.$$

By excision we have

$$H^{q+1}(E(D), \partial E(D); \mathbf{Z}) \simeq H^{q+1}(W, N(D) \cup \partial W; \mathbf{Z}).$$

Remember that  $W$  is obtained from  $D^{n+3}$  by  $(n+1)/2$ -surgery. It implies that

$$\tilde{H}^i(W; \mathbf{Z}) = 0 \quad \text{if } i \neq (n+1)/2 + 1.$$

In particular

$$\tilde{H}^i(W; \mathbf{Z}) = 0 \quad \text{for } i \leq 3$$

as  $n \geq 5$ . Therefore it follows from the exact sequence of the pair  $(W, N(D) \cup \partial W)$  that

$$H^{q+1}(W, N(D) \cup \partial W; \mathbf{Z}) \simeq \tilde{H}^q(N(D) \cup \partial W; \mathbf{Z}) \quad \text{for } q \leq 2.$$

Here the Mayer-Vietoris exact sequence of the triad  $(N(D) \cup \partial W; N(D), \partial W)$  shows that

$$\tilde{H}^q(N(D) \cup \partial W; \mathbf{Z}) = 0 \quad \text{for } q = 0, 1,$$

because  $N(D)$  is contractible,  $\partial W = S^{n+2}$ , and  $N(D) \cap \partial W = S^n \times S^1$ . Hence (4.8) is satisfied, and we have obtained the desired map  $h$ .

*Step 3.* Since  $W$  is framed, the framing of the stable normal bundle  $v(W)$  of  $W$  induces a stable bundle map  $b: v(W) \rightarrow v(D^{n+3})$  which covers  $h$ . The triple  $\mathcal{B} = (W, h, b)$  is called a normal map.

The identity map  $Id: (M, L) \times I \rightarrow (M, L) \times I$  gives a normal map where the stable bundle map is also the identity. We shall denote the normal

map by  $\mathcal{B}_{Id} = ((M, L) \times I, Id, Id)$ . The maps  $h$  and  $Id$  are both diffeomorphisms on  $N(D)$  and  $N(L \times I)$  respectively; so one can do the boundary connected sum of  $\mathcal{B}$  and  $\mathcal{B}_{Id}$  at points of  $K$  and  $L \times \{1\}$ . This yields a new normal map  $\mathcal{B}_{Id} \natural \mathcal{B} = (M \times I \natural W, Id \natural h, Id \natural b)$ . Here we naturally identify the target space  $(M, L) \times I \natural (D^{n+3}, D^{n+1})$  with  $(M, L) \times I$ . Since  $Id \natural h$  is a diffeomorphism on  $N(L \times I \natural D)$ , it gives a product structure on  $N(L \times I \natural D)$ . Thus we get a cobordism  $E(L \times I \natural D)$  relative boundary between  $E(L \# K)$  and  $E(L)$ .

*Step 4.*  $Id \natural h|_{E(L)}: E(L) \rightarrow E(L) \times \{0\}$  (the 0-level) is the identity; so it is a simple homotopy equivalence. We shall observe that  $h_1 = Id \natural h|_{E(L \# K)}: E(L \# K) \rightarrow E(L) \times \{1\}$  (the 1-level) is also a simple homotopy equivalence.

We have a decomposition

$$E(L \# K) = E(L) \cup E(K)$$

in the same sense as (3.7). Hence, similarly to (3.8) one can see

$$(4.9) \quad \pi_1(E(L \# K)) \simeq \pi_1(E(L))$$

where the inclusion map induces the isomorphism.

We can view  $E(L) \times \{1\}$  as  $E(L \# S^n)$  and we also have

$$E(L \# S^n) = E(L) \cup E(S^n).$$

Then the map  $h_1$  can be viewed as the identity on  $E(L)$  and  $h$  on  $E(K)$ . This together with (4.9) shows that  $h_{1*}: \pi_1(E(L \# K)) \rightarrow \pi_1(E(L \# S^n))$  is an isomorphism.

As before we consider the map  $\tilde{h}_1: \tilde{E}(L \# K) \rightarrow \tilde{E}(L \# S^n)$  lifted to the universal covers. Since  $\langle m \rangle = 1$ , we have a diagram

$$(4.10) \quad \begin{array}{ccc} \tilde{E}(L \# K) & = & \tilde{E}(L) \cup E(K) \times \Pi \\ \tilde{h}_1 \downarrow & & \downarrow Id & \downarrow h|_{E(K)} \times Id \\ \tilde{E}(L \# S^n) & = & \tilde{E}(L) \cup E(S^n) \times \Pi, \end{array}$$

where  $\Pi = \pi_1(M - L)$  as before. Since  $h|_{E(K)}$  is a homology equivalence, the above diagram tells us that  $\tilde{h}_{1*}: H_q(\tilde{E}(L \# K); \mathbf{Z}) \rightarrow H_q(\tilde{E}(L \# S^n); \mathbf{Z})$  is an isomorphism as  $\mathbf{Z}[\Pi]$ -modules. Therefore  $h_1$  is a homotopy equivalence by the same reason as before.

The assumption  $\langle m \rangle = 1$  together with the above diagram tells us that  $\tau(h_1) \in Wh(\Pi)$  comes from an element of  $Wh(1)$ . Hence  $\tau(h_1) = 0$  as  $Wh(1) = 0$ .

Step 5. By step 4  $\bar{h} = Id \# h|_{E(L \times I \# D)}: E(L \times I \# D) \rightarrow E(L \times I \# D^{n+1}) = E(L \times I)$  is a simple homotopy equivalence on the boundary. We convert  $\bar{h}$  into a simple homotopy equivalence by surgery without touching the boundary. The obstruction  $\sigma(\bar{h})$  lies in an  $L$ -group  $L_{n+3}(\Pi, 1)$  where 1 denotes the trivial homomorphism from  $\Pi$  to  $\mathbf{Z}_2$  (note, since  $M$  is oriented and hence so is  $E(L \times I)$ , the orientation homomorphism:  $\Pi = \pi_1(E(L \times I)) \rightarrow \mathbf{Z}_2$  is trivial).

We have a diagram similar to (4.10):

$$\begin{array}{ccc} E(L \times I \# D) & = & E(L \times I) \cup E(D) \\ \bar{h} \downarrow & & \downarrow Id & \downarrow h \\ E(L \times I \# D^{n+1}) & = & E(L \times I) \cup E(D^{n+1}). \end{array}$$

The surgery obstruction  $\sigma(h)$  to converting  $h$  to a simple homotopy equivalence by surgery without touching the boundary lies in  $L_{n+3}(\mathbf{Z}, 1)$  because  $\pi_1(E(D^{n+1}))$  is isomorphic to  $\mathbf{Z}$ . The above diagram together with the assumption  $\langle m \rangle = 1$  tells us that

$$\sigma(\bar{h}) = \beta_* \alpha_* \sigma(h)$$

where  $\alpha_*: L_{n+3}(\mathbf{Z}, 1) \rightarrow L_{n+3}(1, 1)$  and  $\beta_*: L_{n+3}(1, 1) \rightarrow L_{n+3}(\Pi, 1)$  are the homomorphisms induced from the trivial homomorphisms  $\alpha: \mathbf{Z} \rightarrow 1$  and  $\beta: 1 \rightarrow \Pi$  respectively. It is well-known that

$$L_{n+3}(1, 1) \simeq \begin{cases} \mathbf{Z} & \text{if } n+3 \equiv 0 \pmod{4}, \\ \mathbf{Z}_2 & \text{if } n+3 \equiv 2 \pmod{4}. \end{cases}$$

As easily observed  $\alpha_* \sigma(h)$  is given by

$$\begin{cases} \text{Sign } W & \text{if } n+3 \equiv 0 \pmod{4} \\ c(W) & \text{if } n+3 \equiv 2 \pmod{4} \end{cases}$$

through the above isomorphism. Remember that  $W$  is framed cobordant to  $D^{n+3}$  relative boundary by the construction. Therefore those invariants vanish and hence  $\sigma(\bar{h}) = 0$ .

Consequently we have obtained a cobordism  $U'$  relative boundary between  $E(L \# K)$  and  $E(L)$  together with a simple homotopy equivalence  $F: U' \rightarrow E(L \times I)$  which is the identity on the 0-level. Let  $i_0: E(L) \rightarrow U'$  and  $j_0: E(L) \rightarrow E(L \times I)$  be the inclusion maps from the 0-level to the cobordisms. Since  $F \circ i_0 = j_0 \circ Id$  where  $Id: E(L) \rightarrow E(L)$  denotes the identity map, we have

$$\tau(F) + F_*\tau(i_0) = \tau(j_0) + j_{0*}\tau(Id)$$

(see [M1, Lemma 7.8]). Here  $F$ ,  $j_0$ , and  $Id$  are all simple homotopy equivalences; so these Whitehead torsions vanish. Hence it follows that  $\tau(i_0) = 0$ , because  $F_*: Wh(\pi_1(U')) \rightarrow Wh(\pi_1(E(L \times I)))$  is an isomorphism. This means that  $U'$  is an  $s$ -cobordism. Therefore  $(S^{n+2}, K) \in I_0(M, L)$  by Lemma 1.6. Q.E.D.

### § 5. TYPE 3 CASE

In this section we treat the case where  $\langle m \rangle$  or  $[m]$  is of order  $p$  ( $p$  is not necessarily a prime number). We begin with

LEMMA 5.1. *Suppose  $[m]$  is of order  $p$ . Then if  $(S^{n+2}, K) \in I(M, L)$ , then  $(S^{n+2}, K)_p$  is a homotopy  $(n+2)$ -sphere.*

*Proof.* Let  $r$  be the order of  $\text{Tor } H_1(M - L; \mathbf{Z})$ , and let  $\gamma$  be the canonical epimorphism  $\pi_1(M - L) \rightarrow H_1(M - L; \mathbf{Z}) \otimes \mathbf{Z}_r$ . Since the order of  $\gamma(\langle m \rangle)$  is  $p$ , we obtain the desired result by an argument similar to the proof of Lemma 2.1. Q.E.D.

If  $p \geq 2$ , there are infinitely many knots  $(S^{n+2}, K)$  such that  $(S^{n+2}, K)_p$  is not a homotopy  $(n+2)$ -sphere; so Lemma 5.1 shows that  $I(M, L) \subsetneq \mathcal{K}_n$  for such  $(M, L)$ .

The rest of this section is devoted to looking for a non-trivial knot in  $I(M, L)$  or  $I_0(M, L)$ . We will extend Proposition 3.6 and 4.2 to the case where  $\langle m \rangle$  is of order  $p$ . Lemma 5.1 reminds us of counterexamples to the generalized Smith conjecture.

Let  $(S^{n+2}, K)$  be an  $n$ -knot which bounds a disk pair  $(D^{n+3}, D)$  such that  $(D^{n+3}, D)_p$  is a homotopy  $(n+3)$ -disk. Since  $(S^{n+2}, K)_p$  is the boundary of  $(D^{n+3}, D)_p$ ,  $(S^{n+2}, K)_p$  is a homotopy  $(n+2)$ -sphere. If  $n+3 \geq 5$ , then  $(D^{n+3}, D)_p$  is diffeomorphic to  $D^{n+3}$  and hence  $(S^{n+2}, K)_p$  is diffeomorphic to  $S^{n+2}$ .

The  $p$ -fold branched cyclic covering  $(D^{n+3}, D)_p$  supports a  $\mathbf{Z}_p$ -action with the branch set  $D$  as the fixed point set. Let  $E(D)_p$  be the exterior of  $D$  in  $(D^{n+3}, D)_p$  and let  $\rho: S^1 \rightarrow E(D)_p$  be an equivariant embedding of a meridian of  $D$  in  $E(D)_p$ , where the standard free  $\mathbf{Z}_p$ -action is considered on  $S^1$ . Since  $\rho$  is a homology equivalence and equivariant, the Whitehead torsion of  $\rho$  is defined in  $Wh(\mathbf{Z}_p)$ . Clearly it is independent of the choice of  $\rho$ ; so we shall denote it by  $\tau_p(D^{n+3}, D)$ .

The following theorem is an extension of Proposition 3.6.

**THEOREM 5.2.** *Suppose  $\langle m \rangle$  is of order  $p$  ( $p$  may be equal to 1) for  $(M^{n+2}, L^n)$  and  $n \geq 4$ . Then  $(S^{n+2}, K) \in I_0(M, L)$  if it bounds a disk pair  $(D^{n+3}, D)$  such that*

- (1)  $(D^{n+3}, D)_p$  is diffeomorphic to  $D^{n+3}$ ,
- (2)  $\mu_* \tau_p(D^{n+3}, D) = 0$ ,

where  $\mu_*: Wh(\mathbf{Z}_p) \rightarrow Wh(\pi_1(M-L))$  is the homomorphism induced from a homomorphism  $\mu: \mathbf{Z}_p \rightarrow \pi_1(M-L)$  sending a generator of  $\mathbf{Z}_p$  to  $\langle m \rangle \in \pi_1(M-L)$ .

**Remark 5.3.** (1) For each  $p$ , there are infinitely many  $n$ -knots satisfying the conditions (1) and (2) in Theorem 5.2. For example the  $\mathbf{Z}_p$ -orbit spaces of Sumners' knots [R, p. 347] (which are counterexamples to the generalized Smith conjecture) are the desired knots. In fact,  $\tau_p(D^{n+3}, D) = 0$  for them.

(2) If  $p = 1, 2, 3, 4$ , or  $6$ , then  $Wh(\mathbf{Z}_p) = 0$ . Hence the condition (2) of Theorem 5.2 is trivially satisfied in these cases.

*Proof of Theorem 5.2.* We shall observe that the proof of Proposition 3.6 works with a little modification. As before  $E(L \times I \natural D)$  can be viewed as a cobordism relative boundary between  $E(L)$  and  $E(L \# K)$ . We shall check that this is an  $s$ -cobordism.

The condition (1) implies that

$$(5.4) \quad \pi_1(E(D)) / \langle m^p \rangle \simeq \mathbf{Z}_p$$

where a meridian of  $D$  in  $D^{n+3}$  is also denoted by  $m$ . Hence it follows from the decomposition (3.7) that

$$\begin{aligned} (5.5) \quad \pi_1(E(L \times I \natural D)) &\simeq \pi_1(E(L \times I)) \underset{\langle m \rangle}{\ast} \pi_1(E(D)) \\ &\simeq \pi_1(E(L \times I)) \underset{\mathbf{Z}_p}{\ast} \pi_1(E(D)) / \langle m^p \rangle \\ &\quad (\text{as } \langle m \rangle \text{ is of order } p \text{ in } \pi_1(E(L \times I))) \\ &\simeq \pi_1(E(L \times I)) \quad (\text{by (5.4)}) \end{aligned}$$

This implies that the inclusion map  $i: E(L) = E(L) \times \{0\} \rightarrow E(L \times I \natural D)$  induces an isomorphism  $\pi_1(E(L)) \rightarrow \pi_1(E(L \times I \natural D))$ .

We consider the map  $\tilde{i}: \tilde{E}(L) \rightarrow \tilde{E}(L \times I \natural D)$  lifted to the universal cover. Let  $q: \tilde{E}(L \times I \natural D) \rightarrow E(L \times I \natural D)$  be the covering projection map. By (5.5)  $q^{-1}(E(L \times I))$  is exactly the universal cover  $\tilde{E}(L \times I)$ . As for  $q^{-1}(E(D))$  we need a little consideration. The above observation (5.5) shows that the image of  $j_*: \pi_1(E(D)) \rightarrow \pi_1(E(L \times I \natural D))$  is isomorphic to  $\mathbf{Z}_p$ , where  $j$  is the inclusion

map. We shall identify  $j_*\pi_1(E(D))$  with  $\mathbf{Z}_p$ . Remember that  $\mathbf{Z}_p$  acts freely on  $E(D)_p$  as covering transformations.

*Claim 5.6.*  $q^{-1}(E(D)) = E(D)_p \times_{\mathbf{Z}_p} \Pi$ , where the right hand side denotes the orbit space of  $E(D)_p \times \Pi$  by the diagonal  $\mathbf{Z}_p$ -action defined by  $s \cdot (x, g) = (xs^{-1}, sg)$  for  $s \in \mathbf{Z}_p$ ,  $x \in E(D)_p$ , and  $g \in \Pi$ .

*Proof.* The  $\Pi$ -covering  $q^{-1}(E(D)) \rightarrow E(D)$  is classified by the map:  $E(D) \rightarrow B\Pi$  induced from the homomorphism  $j_*: \pi_1(E(D)) \rightarrow \Pi = \pi_1(E(L \times I \pitchfork D))$ . Here  $j_*$  factors through the inclusion  $\kappa: \mathbf{Z}_p \rightarrow \Pi$ :

$$\begin{array}{ccc} \pi_1(E(D)) & \xrightarrow{j_*} & \Pi \\ \ell \searrow & & \nearrow \kappa \\ & \cup & \\ \mathbf{Z}_p & & \end{array}$$

The pullback of the universal  $\Pi$ -bundle  $E\Pi \rightarrow B\Pi$  by  $\kappa$  is of the form  $E\mathbf{Z}_p \times_{\mathbf{Z}_p} \Pi \rightarrow B\mathbf{Z}_p$ . In fact, since  $E\mathbf{Z}_p = E\Pi$ , the map  $(u, g) \rightarrow ug$  ( $u \in E\mathbf{Z}_p$ ,  $g \in \Pi$ ) is defined from  $E\mathbf{Z}_p \times_{\mathbf{Z}_p} \Pi$  to  $E\Pi$ . The map induces a  $\Pi$ -bundle map from  $E\mathbf{Z}_p \times_{\mathbf{Z}_p} \Pi \rightarrow B\Pi$  to  $E\Pi \rightarrow B\Pi$ . On the other hand the covering induced from the homomorphism  $\ell: \pi_1(E(D)) \rightarrow \mathbf{Z}_p$  is exactly the  $\mathbf{Z}_p$ -covering  $E(D)_p \rightarrow E(D)$ . These prove the claim.

Consequently we have a decomposition

$$(5.7) \quad \tilde{E}(L \times I \pitchfork D) = \tilde{E}(L \times I) \cup E(D)_p \times_{\mathbf{Z}_p} \Pi,$$

where  $\tilde{E}(L \times I)$  and  $E(D)_p \times_{\mathbf{Z}_p} \Pi$  are pasted together along  $D^n \times S^1 \times_{\mathbf{Z}_p} \Pi$  equivariantly embedded in their boundaries. The condition (1) means that  $E(D)_p$  is a homology circle. This together with (5.7) tells us that  $\tilde{i}: \tilde{E}(L \times I) \rightarrow \tilde{E}(L \times I \pitchfork D)$  induces an isomorphism on homology as  $\mathbf{Z}[\Pi]$ -modules. Hence  $i$  is a homotopy equivalence.

The decomposition (5.7) also tells us that

$$\tau(i) = \mu_*\tau_p(D^{n+3}, D) \quad \text{up to sign}.$$

Hence  $\tau(i) = 0$  by the condition (2). Therefore  $E(L \times I \pitchfork D)$  is an  $s$ -cobordism relative boundary. The theorem then follows from Lemma 1.6. Q.E.D.

A torsion  $\tau_p(S^{n+2}, K)$  is defined similarly to  $\tau_p(D^{n+3}, D)$  if  $(S^{n+2}, K)_p$  is a homotopy  $(n+2)$ -sphere. The following theorem is an extension of Proposition 4.2.

**THEOREM 5.8.** *Suppose  $\langle m \rangle$  is of order  $p$  ( $p$  may be equal to 1) for  $(M^{n+2}, L^n)$  and  $n \geq 4$ . Let  $a_{n,p} = 2$  if  $n \equiv 0 \pmod{4}$  and  $p$  is even, and let  $a_{n,p} = 1$  otherwise. Then  $a_{n,p}(S^{n+2}, K) \in I_0(M, L)$  if*

- (1)  $\sigma(S^{n+2}, K) = 0$  in case  $n$  is odd.
- (2)  $(S^{n+2}, K)_p$  is a homotopy  $(n+2)$ -sphere,
- (3)  $a_{n,p}\mu_*\tau_p(S^{n+2}, K) = 0$

where  $\mu_*$  is the same as in Theorem 5.2.

*Proof.* The argument developed in Steps 1, 2, and 3 of the proof of Proposition 4.2 still works. Step 4 needs a little modification. Instead of (4.10) we have

$$(5.9) \quad \begin{aligned} \tilde{E}(L \# K) &= \tilde{E}(L) \cup E(K)_p \times_{\mathbf{Z}_p} \Pi \\ \tilde{h}_1 \downarrow &\quad \downarrow Id & \downarrow h_p \times Id \\ \tilde{E}(L \# S^n) &= \tilde{E}(L) \cup E(S^n)_p \times_{\mathbf{Z}_p} \Pi \end{aligned} ,$$

(see (5.7)) where  $h_p: E(K)_p \rightarrow E(S^n)_p$  denotes the lifting of  $h$  to the  $\mathbf{Z}_p$ -covers. Since  $h_p$  is a homology equivalence, the above diagram tells us that  $\tilde{h}_1$  is a homotopy equivalence.

It also tells us that

$$\tau(h_1) = -\mu_*\tau_p(S^{n+2}, K),$$

which vanishes by the condition (3). Hence  $h_1: E(L \# K) \rightarrow E(L \# S^n)$  is a simple homotopy equivalence.

Step 5 also needs some modification. We need to replace  $\alpha$  and  $\beta$  by the canonical epimorphism  $\gamma: \mathbf{Z} \rightarrow \mathbf{Z}_p$  and  $\mu: \mathbf{Z}_p \rightarrow \Pi$  respectively. Then we have

$$\sigma(\bar{h}) = \mu_*\gamma_*\sigma(h).$$

Here we distinguish three cases to observe the value  $\sigma(\bar{h})$ .

*Case 1.* The case where  $n$  is odd. In this case the trivial homomorphism  $\alpha: \mathbf{Z} \rightarrow 1$  induces an isomorphism  $L_{n+3}(\mathbf{Z}, 1) \rightarrow L_{n+3}(1, 1)$  ([W11, 13A.8]). As observed in Step 5 of the proof of Proposition 4.2,  $\alpha_*(\sigma(h))$  vanishes. Hence  $\sigma(h) = 0$ , so  $\sigma(\bar{h}) = 0$ .

*Case 2.* The case where  $n \equiv 2 \pmod{4}$  or  $p$  is odd. According to Wall [W12] or Bak [Ba],  $L_{n+3}(\mathbf{Z}_p, 1) = 0$  in this case. Since  $\gamma_*\sigma(h)$  lies in  $L_{n+3}(\mathbf{Z}_p, 1)$ ,  $\gamma_*\sigma(h) = 0$  and hence  $\sigma(\bar{h}) = 0$ .

Case 3. The case where  $n \equiv 0 \pmod{4}$  and  $p$  is even. In this case  $L_{n+3}(\mathbf{Z}_p, 1) \simeq \mathbf{Z}_2$ . Since the value  $\gamma_* \sigma(h) \in L_{n+3}(\mathbf{Z}_p, 1)$  is additive with respect to connected sum, it necessarily vanishes for  $(S^{n+2}, K) \# (S^{n+2}, K)$ .

The rest of the argument is the same as that in Step 5. This proves the theorem. Q.E.D.

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