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Autor: Choudhry, Ajai

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REPRESENTATION OF EVERY RATIONAL NUMBER AS AN ALGEBRAIC SUM OF FIFTH POWERS OF RATIONAL NUMBERS

by Ajai CHOUDHRY

The representation of a rational number as an algebraic sum of k^{th} powers of rational numbers has been considered by Subba Rao [1]. Let $g_1(k)$ be defined as the least integer s such that every rational number r can be expressed in the form

$$r = a_1 x_1^k + a_2 x_2^k + \dots + a_s x_s^k \tag{1}$$

where $a_i = \pm 1$ and all of the values of x_i are rational. It has been shown [1] that

$$g_1(5) \leqslant 8 .$$

In this note we shall prove that $g_1(5) \leqslant 6$. We shall also obtain a parametric solution of the Diophantine equation

$$x_1^5 + x_2^5 + \dots + x_s^5 = 0 \quad \text{for} \quad s \geqslant 6 . \tag{2}$$

The result follows from the identity

$$\begin{aligned} &(t^6x - t^{15} - t^5 - 1)^5 + (t^3x + t^2)^5 + (x - t^{14})^5 \\ &- (t^6x - t^{15} - t^5 + 1)^5 - (t^3x - 2t^{12} - t^2)^5 - (x + t^{14})^5 \\ = &- 2(t^{40} + t^{30} - t^{10} - 1)(20t^{11}x + t^{30} - 12t^{20} - 8t^{10} - 1) . \end{aligned} \tag{3}$$

As the expression on the right-hand side of (3) is linear in x, it can be equated to any rational number r, and solved for x, which leads to the result

$$g_1(5) \leqslant 6 .$$

Thus, every rational number can be expressed as the algebraic sum of at most six fifth powers of rational numbers, and since in the above discussion, t can be taken as any rational number except that $t \neq 0, \pm 1$, this can be done in infinitely many ways.

To solve (2) for $s = 6$, we equate the right-hand side of (3) to 0, and obtain the result

$$\begin{aligned}&(t^{36}+8t^{26}+12t^{16}+20t^{11}-t^{6})^5 + (t^{33}-12t^{23}-28t^{13}-t^{3})^5\\ &+ (t^{30}+20t^{25}-12t^{20}-8t^{10}-1)^5 - (t^{36}+8t^{26}+12t^{16}-20t^{11}-t^{6})^5\\ &- (t^{33}+28t^{23}+12t^{13}-t^{3})^5 - (t^{30}-20t^{25}-12t^{20}-8t^{10}-1)^5 = 0\,.\end{aligned}$$

This result has earlier been given by Moessner [2].

To solve (2) for $s = 6 + m$, $m > 0$, we simply equate the right hand side of (3) to $\sum_{i=1}^{m} x_i^5$ where x_i are any rational numbers, and solve for x, which leads to a solution of (2) for $s > 6$. Solutions in integers can be obtained by multiplying by a suitable constant.

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Ajai Choudhry

Embassy of India
Rejtana 15
Warsaw (Poland)