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In order to calculate  $\text{Ind}(\gamma; s)$  we replace  $U$  by a small pointed neighbourhood  $D^*$  of  $s$ . With the notation of (7.2) let us write  $\rho = \rho(s)\rho_s$  and deduce that

$$\langle \partial\gamma, \omega \rangle = \rho(s)\text{Tr}(\omega; s), \quad \omega \in \Gamma(D^*, \Omega^{n-1}), \quad d\omega = 0.$$

We can now conclude from (7.6) that

$$\text{Ind}(\gamma; s) = \rho(s), \quad s \in X - U.$$

This reveals that  $s \mapsto \text{Ind}(\gamma; s)$  is a compactly supported, locally constant function on  $X - U$ .

For a given fixed point  $s \notin \text{Supp}(b\gamma)$  choose  $U$  to be an open neighbourhood of  $\text{Supp}(b\gamma)$  with  $\bar{U}$  compact and  $s \notin U$ . We can apply the considerations above and conclude that the winding number is constant in a neighbourhood of  $s$  and zero outside some compact neighbourhood of  $\text{Supp}(b\gamma)$ . Q.E.D.

(7.11) COROLLARY. *Let  $\gamma$  be a compact  $n$ -chain on the oriented smooth manifold  $X$  and  $U$  an open subset of  $X$  containing  $\text{Supp}(b\gamma)$ . The relative de Rham homology class*

$$[\gamma] \in H_n^c(X, U; \mathbb{C})$$

is zero if and only if  $\text{Ind}(\gamma; s) = 0$  for all  $s \in X - U$ .

*Proof.* This is a corollary to the proof of (7.10) rather than the statement (7.10). Anyway, the basic point is Poincaré duality (6.6). Q.E.D.

## 8. CAUCHY'S RESIDUE THEOREM

We shall consider a smooth map  $\gamma: S^{n-1} \rightarrow E$  from the oriented  $n - 1$  sphere into an oriented  $n$ -dimensional real vector space  $E$ . For a point  $s$  outside  $\gamma(S^{n-1})$  pick a closed  $(n - 1)$ -form  $\omega_s$  on  $E - \{s\}$  with  $\text{Tr}(\omega_s; s) = 1$  and define the *winding number* of  $\gamma$  with respect to  $s$  to be

$$(8.1) \quad \text{Ind}(\gamma; s) = \int_{S^{n-1}} \gamma^* \omega_s.$$

(8.2) CAUCHY'S RESIDUE THEOREM. *Let  $\gamma: S^{n-1} \rightarrow X$  denote a smooth map into an open subset  $X$  of  $E$  with  $\text{Ind}(\gamma; z) = 0$  for all  $z \in E - X$ .*

For a closed and discrete subset  $S$  of  $X$  disjoint from  $\gamma(S^{n-1})$  only finitely many of the numbers  $\text{Ind}(\gamma; s), s \in S$ , are distinct from zero and

$$\int_{S^{n-1}} \gamma^* \omega = \sum_{s \in S} \text{Ind}(\gamma; s) \text{Tr}(\omega; s)$$

for any closed form  $\omega$  on  $X - S$ .

*Proof.* The long exact de Rham homology sequence for the pair  $X - S, E$  degenerates into an isomorphism

$$b: H_n^c(E, X - S; \mathbf{C}) \xrightarrow{\sim} H_{n-1}^c(X - S, \mathbf{C}).$$

Let us view  $\gamma$  as a homology class on  $X - S$  and introduce the class

$$b^{-1}\gamma \in H_n^c(E, X - S; \mathbf{C}).$$

Let us notice that the winding number (8.1) and (7.8) agree. Thus we conclude from (7.11) that  $b^{-1}\gamma$  maps to zero in  $H_n^c(E, X; \mathbf{C})$  and consequently that  $\gamma$  is homologous to zero on  $X$ . The exact sequence

$$0 \rightarrow H_n^c(X, X - S; \mathbf{C}) \xrightarrow{b} H_{n-1}^c(X - S, \mathbf{C}) \rightarrow H_{n-1}^c(X, \mathbf{C})$$

allows us to interpret  $\gamma$  as a relative class

$$\gamma \in H_n^c(X, X - S; \mathbf{C}).$$

The class  $\gamma$  can be specified by the formula

$$\langle b\gamma, \omega \rangle = \int_{S^{n-1}} \gamma^* \omega, \quad \omega \in \Gamma(X - S, \Omega^{n-1}), \quad d\omega = 0.$$

From the decomposition (4.9) and excision (4.6) we deduce a canonical isomorphism

$$H_n(X, X - S; \mathbf{C}) \xrightarrow{\sim} \bigoplus_{s \in S} H_n(X, X - \{s\}; \mathbf{C})$$

which allow us to decompose the class  $\gamma$  into a finite sum, compare (7.6),

$$\gamma = \sum_{s \in S} \text{Ind}(\gamma; s) \theta_s.$$

Using the general Stokes formula (5.3) we get that

$$\langle b\gamma, \omega \rangle = \langle \gamma, \partial\omega \rangle = \sum \text{Ind}(\gamma; s) \langle \theta_s, \partial\omega \rangle$$

and the result follows from formula (7.6).

Q.E.D.