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In order to calculate $\text{Ind}(\gamma; s)$ we replace U by a small pointed neighbourhood D^* of s . With the notation of (7.2) let us write $\rho = \rho(s)\rho_s$ and deduce that

$$\langle \partial\gamma, \omega \rangle = \rho(s)\text{Tr}(\omega; s), \quad \omega \in \Gamma(D^*, \Omega^{n-1}), \quad d\omega = 0.$$

We can now conclude from (7.6) that

$$\text{Ind}(\gamma; s) = \rho(s), \quad s \in X - U.$$

This reveals that $s \mapsto \text{Ind}(\gamma; s)$ is a compactly supported, locally constant function on $X - U$.

For a given fixed point $s \notin \text{Supp}(b\gamma)$ choose U to be an open neighbourhood of $\text{Supp}(b\gamma)$ with \bar{U} compact and $s \notin U$. We can apply the considerations above and conclude that the winding number is constant in a neighbourhood of s and zero outside some compact neighbourhood of $\text{Supp}(b\gamma)$. Q.E.D.

(7.11) COROLLARY. *Let γ be a compact n -chain on the oriented smooth manifold X and U an open subset of X containing $\text{Supp}(b\gamma)$. The relative de Rham homology class*

$$[\gamma] \in H_n^c(X, U; \mathbb{C})$$

is zero if and only if $\text{Ind}(\gamma; s) = 0$ for all $s \in X - U$.

Proof. This is a corollary to the proof of (7.10) rather than the statement (7.10). Anyway, the basic point is Poincaré duality (6.6). Q.E.D.

8. CAUCHY'S RESIDUE THEOREM

We shall consider a smooth map $\gamma: S^{n-1} \rightarrow E$ from the oriented $n - 1$ sphere into an oriented n -dimensional real vector space E . For a point s outside $\gamma(S^{n-1})$ pick a closed $(n - 1)$ -form ω_s on $E - \{s\}$ with $\text{Tr}(\omega_s; s) = 1$ and define the *winding number* of γ with respect to s to be

$$(8.1) \quad \text{Ind}(\gamma; s) = \int_{S^{n-1}} \gamma^*\omega_s.$$

(8.2) CAUCHY'S RESIDUE THEOREM. *Let $\gamma: S^{n-1} \rightarrow X$ denote a smooth map into an open subset X of E with $\text{Ind}(\gamma; z) = 0$ for all $z \in E - X$.*

For a closed and discrete subset S of X disjoint from $\gamma(S^{n-1})$ only finitely many of the numbers $\text{Ind}(\gamma; s), s \in S$, are distinct from zero and

$$\int_{S^{n-1}} \gamma^* \omega = \sum_{s \in S} \text{Ind}(\gamma; s) \text{Tr}(\omega; s)$$

for any closed form ω on $X - S$.

Proof. The long exact de Rham homology sequence for the pair $X - S, E$ degenerates into an isomorphism

$$b: H_n^c(E, X - S; \mathbf{C}) \xrightarrow{\sim} H_{n-1}^c(X - S, \mathbf{C}).$$

Let us view γ as a homology class on $X - S$ and introduce the class

$$b^{-1}\gamma \in H_n^c(E, X - S; \mathbf{C}).$$

Let us notice that the winding number (8.1) and (7.8) agree. Thus we conclude from (7.11) that $b^{-1}\gamma$ maps to zero in $H_n^c(E, X; \mathbf{C})$ and consequently that γ is homologous to zero on X . The exact sequence

$$0 \rightarrow H_n^c(X, X - S; \mathbf{C}) \xrightarrow{b} H_{n-1}^c(X - S, \mathbf{C}) \rightarrow H_{n-1}^c(X, \mathbf{C})$$

allows us to interpret γ as a relative class

$$\gamma \in H_n^c(X, X - S; \mathbf{C}).$$

The class γ can be specified by the formula

$$\langle b\gamma, \omega \rangle = \int_{S^{n-1}} \gamma^* \omega, \quad \omega \in \Gamma(X - S, \Omega^{n-1}), \quad d\omega = 0.$$

From the decomposition (4.9) and excision (4.6) we deduce a canonical isomorphism

$$H_n(X, X - S; \mathbf{C}) \xrightarrow{\sim} \bigoplus_{s \in S} H_n(X, X - \{s\}; \mathbf{C})$$

which allow us to decompose the class γ into a finite sum, compare (7.6),

$$\gamma = \sum_{s \in S} \text{Ind}(\gamma; s) \theta_s.$$

Using the general Stokes formula (5.3) we get that

$$\langle b\gamma, \omega \rangle = \langle \gamma, \partial\omega \rangle = \sum \text{Ind}(\gamma; s) \langle \theta_s, \partial\omega \rangle$$

and the result follows from formula (7.6).

Q.E.D.