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Autor: Iversen, Birger
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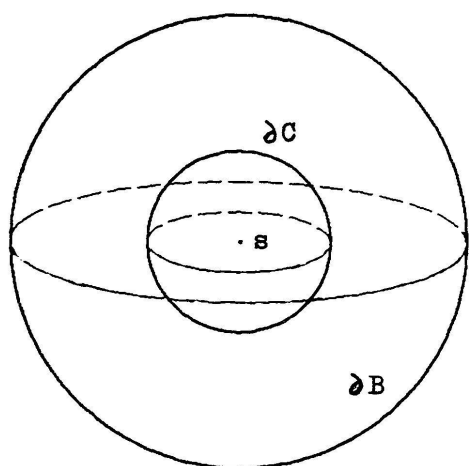
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7. WINDING NUMBERS

Let X be an n -dimensional oriented smooth manifold and s a point of X . Consider a compact n -dimensional submanifold with boundary B with s as an interior point and put

$$(7.1) \quad \text{Tr}(\omega; s) = \int_{\partial B} \omega, \quad \omega \in \Gamma(X - \{s\}, \Omega^{n-1}), d\omega = 0.$$

This symbol is independent of B as it follows by considering a small "ball" C around s contained in the interior of B



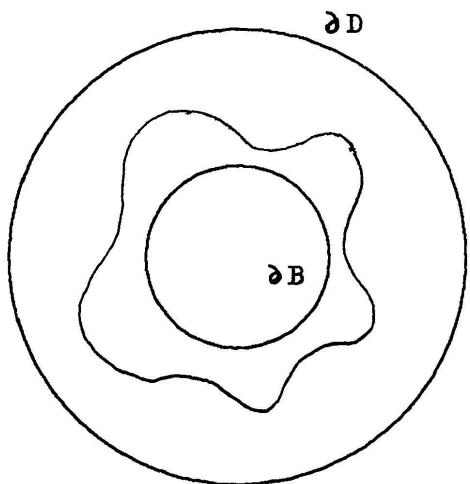
Stokes formula for $B - C^0$

$$\int_{\partial B} \omega - \int_{\partial C} \omega = \int_{B - C^0} d\omega.$$

Alternatively, choose a compactly supported smooth real function ρ_s on X which is constant 1 in a neighbourhood of s . Then

$$(7.2) \quad \text{Tr}(\omega; s) = (-1)^n \int_X \omega \wedge d\rho_s, \quad \omega \in \Gamma(X - \{s\}, \Omega^{n-1}), d\omega = 0.$$

Proof. Choose "balls" B and D with center s such that ρ_s is constant 1 on B while $\text{Supp}(\rho_s)$ is contained in the interior of D . From Stokes formula we get that



$$\begin{aligned} \int_{\partial D} \rho_s \omega - \int_{\partial B} \rho_s \omega &= \int_{D-B} \rho_s \wedge \omega - \int_{D-B} \rho_s \omega \\ &= - \int_{\partial B} \omega = -(-1)^n \int_X \omega \wedge d\rho_s + \int_{D-B} \rho_s d\omega. \end{aligned}$$

Notice that the last terms vanishes in case ω is exact.

Q.E.D.

(7.3) *Example.* Let E denote an oriented n -dimensional Euclidian space. The distance r to the origin defines a 1-form dr^{2-n} on $E - \{0\}$. The dual form $*dr^{2-n}$ in the sense of Hodge is closed with

$$\text{Tr}(*dr^{2-n}; 0) = (2-n)\sigma_{n-1}$$

where σ_{n-1} denotes the area of the unit sphere in E , compare [3] VII. 1.

Let us interpret (7.1) in terms of de Rham homology. Integration of n -forms on X over the manifold B determines a compact n -chain on X whose boundary, as written in (7.1), has support in $X - \{s\}$. The corresponding relative homology class

$$(7.4) \quad \theta_s \in H_n^c(X, X - \{s\}; \mathbf{C}), \quad s \in X,$$

is independent of B : with the notation above, the compact n -chain $\int_B - \int_C$ has support in $X - \{s\}$. The relative homology class we have just constructed is often called *the local orientation class*.

(7.5) **PROPOSITION.** *Let s be a point of an oriented n -dimensional smooth manifold X . The local orientation class θ_s generates $H_n^c(X, X - \{s\}; \mathbf{C})$.*

Proof. With the terminology from section 5 we may express formula (7.1) by means of the local orientation class

$$(7.6) \quad \text{Tr}(\omega; s) = \langle \theta_s, \partial\omega \rangle = \langle b\theta_s, \omega \rangle, \quad \omega \in H^{n-1}(X - \{s\}, \mathbf{C}).$$

In case $n > 2$ we conclude from (7.3), that $\theta_s \neq 0$. The case $n = 2$ is left with the reader.

Q.E.D.

Let us remark that formula (7.2) shows how to identify θ_s under relative Poincaré duality (6.6).

(7.7) **PROPOSITION.** *Let S be a finite subset of the oriented n -dimensional compact manifold X . For any closed form $\omega \in \Gamma(X - S, \Omega^n)$ we have that*

$$\sum_{s \in S} \text{Tr}(\omega; s) = 0.$$

Proof. Let the fundamental class $\theta \in H_n(X, \mathbf{C})$ be given by

$$\langle \theta, \omega \rangle = \int_X \omega, \quad \omega \in \Gamma(X, \Omega^n).$$

Let us consider a point $s \in S$ and use the notation from (7.1). The difference $\int_X - \int_B$ has support in $X - \{s\}$, which shows that the image of θ in $H_n(X, X - \{s\}; \mathbb{C})$ is θ_s . We have that

$$\sum_{s \in S} \text{Tr}(\omega; s) = \sum_{s \in S} \langle \theta_s, \partial \omega \rangle = \langle \theta_S, \partial \omega \rangle = \langle b\theta_S, \omega \rangle$$

where θ_S denotes the restriction of θ to $H_n(X, X - S; \mathbb{C})$. Conclusion by the fact that $b\theta_S = 0$. Q.E.D.

(7.8) *Definition.* Let γ be a compact n -chain on the oriented n -dimensional smooth manifold X . For a point $s \in X$ outside $\text{Supp}(b\gamma)$ the class of γ in $H_n^c(X, X - \{s\}; \mathbb{C})$ can be written

$$[\gamma] = \text{Ind}(\gamma; s)\theta_s, \quad \text{Ind}(\gamma; s) \in \mathbb{C}.$$

The number $\text{Ind}(\gamma; s)$ is called the *winding number* of γ with respect to s .

(7.9) *Example.* Let K denote an n -dimensional compact submanifold with boundary. Integration over K defines a compact n -chain κ with $\text{Supp}(\partial\kappa) = \partial K$. The winding number for κ is 1 in the interior of K and 0 outside K .

(7.10) *THEOREM.* Let γ be a compact n -chain on the oriented n -dimensional smooth manifold X . The winding number $s \mapsto \text{Ind}(\gamma; s)$ is a locally constant function on the complement of $\text{Supp}(b\gamma)$ in X . This function is zero outside some compact subset of X containing $\text{Supp}(b\gamma)$.

Proof. Let us consider an arbitrary open subset U of X containing $\text{Supp}(b\gamma)$. We shall now use relative Poincaré duality to describe the class of γ in $H_n^c(X, U; \mathbb{C})$. According to (6.6) and (6.7) we can represent γ by a relative n -chain of the form

$$\langle \gamma, v \rangle = \int_X \rho v, \quad v \in \Gamma(X, \Omega^n)$$

where ρ is a compactly supported smooth function on X , constant in a neighbourhood of any point s of $Z = X - U$. Let us notice that

$$\langle \partial\gamma, \omega \rangle = (-1)^n \int \omega \wedge d\rho, \quad \omega \in \Gamma(U, \Omega^{n-1}), \quad d\omega = 0.$$

In order to calculate $\text{Ind}(\gamma; s)$ we replace U by a small pointed neighbourhood D^* of s . With the notation of (7.2) let us write $\rho = \rho(s)\rho_s$ and deduce that

$$\langle \partial\gamma, \omega \rangle = \rho(s)\text{Tr}(\omega; s), \quad \omega \in \Gamma(D^*, \Omega^{n-1}), \quad d\omega = 0.$$

We can now conclude from (7.6) that

$$\text{Ind}(\gamma; s) = \rho(s), \quad s \in X - U.$$

This reveals that $s \mapsto \text{Ind}(\gamma; s)$ is a compactly supported, locally constant function on $X - U$.

For a given fixed point $s \notin \text{Supp}(b\gamma)$ choose U to be an open neighbourhood of $\text{Supp}(b\gamma)$ with \bar{U} compact and $s \notin U$. We can apply the considerations above and conclude that the winding number is constant in a neighbourhood of s and zero outside some compact neighbourhood of $\text{Supp}(b\gamma)$. Q.E.D.

(7.11) COROLLARY. *Let γ be a compact n -chain on the oriented smooth manifold X and U an open subset of X containing $\text{Supp}(b\gamma)$. The relative de Rham homology class*

$$[\gamma] \in H_n^c(X, U; \mathbb{C})$$

is zero if and only if $\text{Ind}(\gamma; s) = 0$ for all $s \in X - U$.

Proof. This is a corollary to the proof of (7.10) rather than the statement (7.10). Anyway, the basic point is Poincaré duality (6.6). Q.E.D.

8. CAUCHY'S RESIDUE THEOREM

We shall consider a smooth map $\gamma: S^{n-1} \rightarrow E$ from the oriented $n-1$ sphere into an oriented n -dimensional real vector space E . For a point s outside $\gamma(S^{n-1})$ pick a closed $(n-1)$ -form ω_s on $E - \{s\}$ with $\text{Tr}(\omega_s; s) = 1$ and define the *winding number* of γ with respect to s to be

$$(8.1) \quad \text{Ind}(\gamma; s) = \int_{S^{n-1}} \gamma^* \omega_s.$$

(8.2) CAUCHY'S RESIDUE THEOREM. *Let $\gamma: S^{n-1} \rightarrow X$ denote a smooth map into an open subset X of E with $\text{Ind}(\gamma; z) = 0$ for all $z \in E - X$.*