

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 35 (1989)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CAUCHY RESIDUES AND DE RHAM HOMOLOGY
Autor: Iversen, Birger
Kapitel: 7. WINDING NUMBERS
DOI: <https://doi.org/10.5169/seals-57358>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 27.04.2026

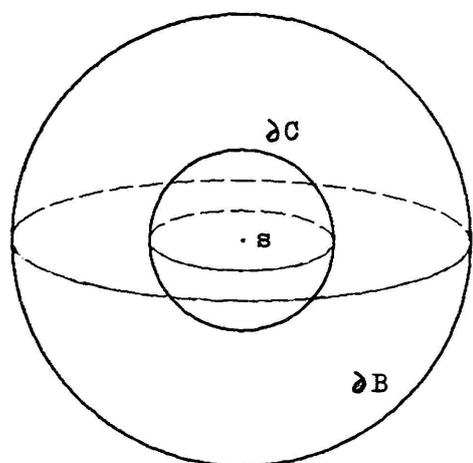
ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

7. WINDING NUMBERS

Let X be an n -dimensional oriented smooth manifold and s a point of X . Consider a compact n -dimensional submanifold with boundary B with s as an interior point and put

$$(7.1) \quad \text{Tr}(\omega; s) = \int_{\partial B} \omega, \quad \omega \in \Gamma(X - \{s\}, \Omega^{n-1}), d\omega = 0.$$

This symbol is independent of B as it follows by considering a small "ball" C around s contained in the interior of B



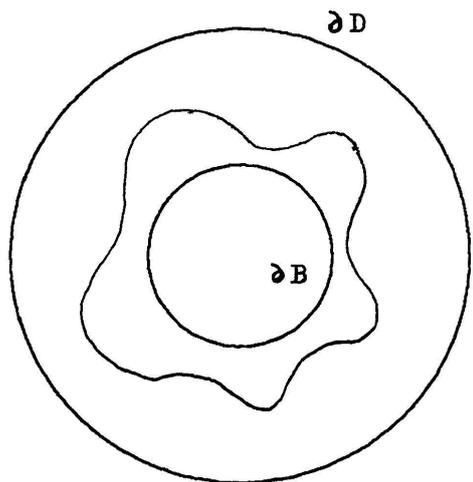
Stokes formula for $B - C^0$

$$\int_{\partial B} \omega - \int_{\partial C} \omega = \int_{B - C^0} d\omega.$$

Alternatively, choose a compactly supported smooth real function ρ_s on X which is constant 1 in a neighbourhood of s . Then

$$(7.2) \quad \text{Tr}(\omega; s) = (-1)^n \int_X \omega \wedge d\rho_s, \quad \omega \in \Gamma(X - \{s\}, \Omega^{n-1}), d\omega = 0.$$

Proof. Choose "balls" B and D with center s such that ρ_s is constant 1 on B while $\text{Supp}(\rho_s)$ is contained in the interior of D . From Stokes formula we get that



$$\begin{aligned} \int_{\partial D} \rho_s \omega - \int_{\partial B} \rho_s \omega &= \int_{D-B} \rho_s \wedge \omega - \int_{D-B} \rho_s \omega \\ &- \int_{\partial B} \omega = -(-1)^n \int_X \omega \wedge d\rho_s + \int_{D-B} \rho_s d\omega. \end{aligned}$$

Notice that the last terms vanishes in case ω is exact.

Q.E.D.

(7.3) *Example.* Let E denote an oriented n -dimensional Euclidian space. The distance r to the origin defines a 1-form dr^{2-n} on $E - \{0\}$. The dual form $*dr^{2-n}$ in the sense of Hodge is closed with

$$\text{Tr}(*dr^{2-n}; 0) = (2-n)\sigma_{n-1}$$

where σ_{n-1} denotes the area of the unit sphere in E , compare [3] VII. 1.

Let us interpret (7.1) in terms of de Rham homology. Integration of n -forms on X over the manifold B determines a compact n -chain on X whose boundary, as written in (7.1), has support in $X - \{s\}$. The corresponding relative homology class

$$(7.4) \quad \theta_s \in H_n^c(X, X - \{s\}; \mathbf{C}), \quad s \in X,$$

is independent of B : with the notation above, the compact n -chain $\int_B - \int_C$ has support in $X - \{s\}$. The relative homology class we have just constructed is often called *the local orientation class*.

(7.5) PROPOSITION. *Let s be a point of an oriented n -dimensional smooth manifold X . The local orientation class θ_s generates $H_n^c(X, X - \{s\}; \mathbf{C})$.*

Proof. With the terminology from section 5 we may express formula (7.1) by means of the local orientation class

$$(7.6) \quad \text{Tr}(\omega; s) = \langle \theta_s, \partial\omega \rangle = \langle b\theta_s, \omega \rangle, \quad \omega \in H^{n-1}(X - \{s\}, \mathbf{C}).$$

In case $n > 2$ we conclude from (7.3), that $\theta_s \neq 0$. The case $n = 2$ is left with the reader.

Q.E.D.

Let us remark that formula (7.2) shows how to identify θ_s under relative Poincaré duality (6.6).

(7.7) PROPOSITION. *Let S be a finite subset of the oriented n -dimensional compact manifold X . For any closed form $\omega \in \Gamma(X - S, \Omega^n)$ we have that*

$$\sum_{s \in S} \text{Tr}(\omega; s) = 0.$$

Proof. Let the *fundamental class* $\theta \in H_n(X, \mathbf{C})$ be given by

$$\langle \theta, \omega \rangle = \int_X \omega, \quad \omega \in \Gamma(X, \Omega^n).$$

Let us consider a point $s \in S$ and use the notation from (7.1). The difference $\int_X - \int_B$ has support in $X - \{s\}$, which shows that the image of θ in $H_n(X, X - \{s\}; \mathbf{C})$ is θ_s . We have that

$$\sum_{s \in S} \text{Tr}(\omega; s) = \sum_{s \in S} \langle \theta_s, \partial\omega \rangle = \langle \theta_S, \partial\omega \rangle = \langle b\theta_S, \omega \rangle$$

where θ_s denotes the restriction of θ to $H_n(X, X - S; \mathbf{C})$. Conclusion by the fact that $b\theta_S = 0$. Q.E.D.

(7.8) *Definition.* Let γ be a compact n -chain on the oriented n -dimensional smooth manifold X . For a point $s \in X$ outside $\text{Supp}(b\gamma)$ the class of γ in $H_n^c(X, X - \{s\}; \mathbf{C})$ can be written

$$[\gamma] = \text{Ind}(\gamma; s)\theta_s, \quad \text{Ind}(\gamma; s) \in \mathbf{C}.$$

The number $\text{Ind}(\gamma; s)$ is called the *winding number* of γ with respect to s .

(7.9) *Example.* Let K denote an n -dimensional compact submanifold with boundary. Integration over K defines a compact n -chain κ with $\text{Supp}(\partial\kappa) = \partial K$. The winding number for κ is 1 in the interior of K and 0 outside K .

(7.10) **THEOREM.** *Let γ be a compact n -chain on the oriented n -dimensional smooth manifold X . The winding number $s \mapsto \text{Ind}(\gamma; s)$ is a locally constant function on the complement of $\text{Supp}(b\gamma)$ in X . This function is zero outside some compact subset of X containing $\text{Supp}(b\gamma)$.*

Proof. Let us consider an arbitrary open subset U of X containing $\text{Supp}(b\gamma)$. We shall now use relative Poincaré duality to describe the class of γ in $H_n^c(X, U; \mathbf{C})$. According to (6.6) and (6.7) we can represent γ by a relative n -chain of the form

$$\langle \gamma, v \rangle = \int_X \rho v, \quad v \in \Gamma(X, \Omega^n)$$

where ρ is a compactly supported smooth function on X , constant in a neighbourhood of any point s of $Z = X - U$. Let us notice that

$$\langle \partial\gamma, \omega \rangle = (-1)^n \int \omega \wedge d\rho, \quad \omega \in \Gamma(U, \Omega^{n-1}), \quad d\omega = 0.$$

In order to calculate $\text{Ind}(\gamma; s)$ we replace U by a small pointed neighbourhood D^* of s . With the notation of (7.2) let us write $\rho = \rho(s)\rho_s$ and deduce that

$$\langle \partial\gamma, \omega \rangle = \rho(s)\text{Tr}(\omega; s), \quad \omega \in \Gamma(D^*, \Omega^{n-1}), \quad d\omega = 0.$$

We can now conclude from (7.6) that

$$\text{Ind}(\gamma; s) = \rho(s), \quad s \in X - U.$$

This reveals that $s \mapsto \text{Ind}(\gamma; s)$ is a compactly supported, locally constant function on $X - U$.

For a given fixed point $s \notin \text{Supp}(b\gamma)$ choose U to be an open neighbourhood of $\text{Supp}(b\gamma)$ with \bar{U} compact and $s \notin U$. We can apply the considerations above and conclude that the winding number is constant in a neighbourhood of s and zero outside some compact neighbourhood of $\text{Supp}(b\gamma)$. Q.E.D.

(7.11) COROLLARY. *Let γ be a compact n -chain on the oriented smooth manifold X and U an open subset of X containing $\text{Supp}(b\gamma)$. The relative de Rham homology class*

$$[\gamma] \in H_n^c(X, U; \mathbb{C})$$

is zero if and only if $\text{Ind}(\gamma; s) = 0$ for all $s \in X - U$.

Proof. This is a corollary to the proof of (7.10) rather than the statement (7.10). Anyway, the basic point is Poincaré duality (6.6). Q.E.D.

8. CAUCHY'S RESIDUE THEOREM

We shall consider a smooth map $\gamma: S^{n-1} \rightarrow E$ from the oriented $n - 1$ sphere into an oriented n -dimensional real vector space E . For a point s outside $\gamma(S^{n-1})$ pick a closed $(n - 1)$ -form ω_s on $E - \{s\}$ with $\text{Tr}(\omega_s; s) = 1$ and define the *winding number* of γ with respect to s to be

$$(8.1) \quad \text{Ind}(\gamma; s) = \int_{S^{n-1}} \gamma^*\omega_s.$$

(8.2) CAUCHY'S RESIDUE THEOREM. *Let $\gamma: S^{n-1} \rightarrow X$ denote a smooth map into an open subset X of E with $\text{Ind}(\gamma; z) = 0$ for all $z \in E - X$.*