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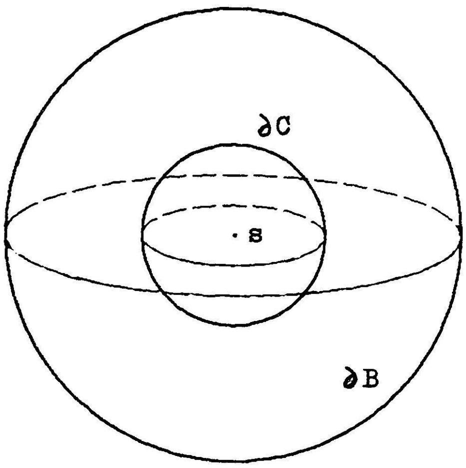
7. Winding numbers

Let X be an n-dimensional oriented smooth manifold and s a point of X. Consider a compact n-dimensional submanifold with boundary B with s as an interior point and put

$$\text{(7.1)} \qquad \operatorname{Tr}(\omega; s) = \int_{\partial B} \omega, \quad \omega \in \Gamma(X-\{s\}, \Omega^{n-1}),\ d\omega = 0.$$

This symbol is independent of B as it follows by considering a small "ball" C around s contained in the interior of B



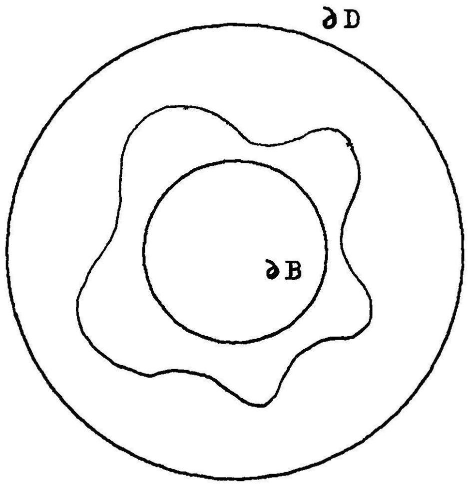
Stokes formula for $B - C^0$

$$\int_{\partial B} \omega - \int_{\partial C} \omega = \int_{B-C^0} d\omega.$$

Alternatively, choose a compactly supported smooth real function ρ_s on X which is constant 1 in a neighbourhood of s. Then

$$\text{(7.2)} \qquad \operatorname{Tr}(\omega; s) = (-1)^n \int_X \omega \wedge d\rho_s, \quad \omega \in \Gamma(X-\{s\}, \Omega^{n-1}),\ d\omega = 0.$$

Proof. Choose "balls" B and D with center s such that ρ_s is constant 1 on B while $\operatorname{Supp}(\rho_s)$ is contained in the interior of D. From Stokes formula we get that



$$\int_{\partial D} \rho_s\omega - \int_{\partial B} \rho_s\omega = \int_{D-B} \rho_s \wedge \omega - \int_{D-B} \rho_s\omega$$

$$-\int_{\partial B} \omega = -(-1)^n \int_X \omega \wedge d\rho_s + \int_{D-B} \rho_s d\omega.$$

Notice that the last terms vanishes in case ω is exact. Q.E.D.

(7.3) *Example.* Let E denote an oriented n-dimensional Euclidian space. The distance r to the origin defines a 1-form dr^{2-n} on $E - \{0\}$. The dual form $*dr^{2-n}$ in the sense of Hodge is closed with

$$\operatorname{Tr}(*dr^{2-n}; 0) = (2-n)\sigma_{n-1}$$

where σ_{n-1} denotes the area of the unit sphere in E, compare [3] VII. 1.

Let us interprete (7.1) in terms of de Rham homology. Integration of n-forms on X over the manifold B determines a compact n-chain on X whose boundary, as written in (7.1), has support in $X - \{s\}$. The corresponding relative homology class

(7.4) $$\theta_s \in H_n^c(X, X-\{s\}; \mathbf{C}), \quad s \in X,$$

is independent of B: with the notation above, the compact n-chain $\int_B - \int_C$ has support in $X - \{s\}$. The relative homology class we have just constructed is often called *the local orientation class.*

(7.5) Proposition. *Let s be a point of an oriented n-dimensional smooth manifold X. The local orientation class θ_s generates $H_n^c(X, X-\{s\}; \mathbf{C})$.*

Proof. With the terminology from section 5 we may express formula (7.1) by means of the local orientation class

(7.6) $$\operatorname{Tr}(\omega; s) = <\theta_s, \partial\omega> = <b\theta_s, \omega>, \quad \omega \in H^{n-1}(X-\{s\}, \mathbf{C}).$$

In case $n > 2$ we conclude from (7.3), that $\theta_s \neq 0$. The case $n = 2$ is left with the reader. Q.E.D.

Let us remark that formula (7.2) shows how to identify θ_s under relative Poincaré duality (6.6).

(7.7) Proposition. *Let S be a finite subset of the oriented n-dimensional compact manifold X. For any closed form $\omega \in \Gamma(X-S, \Omega^n)$ we have that*

$$\sum_{s\in S} \operatorname{Tr}(\omega; s) = 0.$$

Proof. Let the *fundamental class* $\theta \in H_n(X, \mathbf{C})$ be given by

$$<\theta, \omega> = \int_X \omega, \quad \omega \in \Gamma(X, \Omega^n).$$

Let us consider a point $s \in S$ and use the notation from (7.1). The difference $\int_X - \int_B$ has support in $X - \{s\}$, which shows that the image of θ in $H_n(X, X-\{s\}; \mathbf{C})$ is θ_s. We have that

$$\sum_{s\in S} \mathrm{Tr}(\omega; s) = \sum_{s\in S} <\theta_s, \partial\omega> = <\theta_S, \partial\omega> = <b\theta_S, \omega>$$

where θ_S denotes the restriction of θ to $H_n(X, X-S; \mathbf{C})$. Conclusion by the fact that $b\theta_S = 0$. Q.E.D.

(7.8) *Definition.* Let γ be a compact n-chain on the oriented n-dimensional smooth manifold X. For a point $s \in X$ outside $\mathrm{Supp}(b\gamma)$ the class of γ in $H_n^c(X, X-\{s\}; \mathbf{C})$ can be written

$$[\gamma] = \mathrm{Ind}(\gamma; s)\theta_s, \quad \mathrm{Ind}(\gamma; s) \in \mathbf{C}.$$

The number $\mathrm{Ind}(\gamma; s)$ is called the *winding number* of γ with respect to s.

(7.9) *Example.* Let K denote an n-dimensional compact submanifold with boundary. Integration over K defines a compact n-chain κ with $\mathrm{Supp}(\partial\kappa) = \partial K$. The winding number for κ is 1 in the interior of K and 0 outside K.

(7.10) THEOREM. *Let γ be a compact n-chain on the oriented n-dimensional smooth manifold X. The winding number $s \mapsto \mathrm{Ind}(\gamma; s)$ is a locally constant function on the complement of $\mathrm{Supp}(b\gamma)$ in X. This function is zero outside some compact subset of X containing $\mathrm{Supp}(b\gamma)$.*

Proof. Let us consider an arbitrary open subset U of X containing $\mathrm{Supp}(b\gamma)$. We shall now use relative Poincaré duality to describe the class of γ in $H_n^c(X, U; \mathbf{C})$. According to (6.6) and (6.7) we can represent γ by a relative n-chain of the form

$$<\gamma, \nu> = \int_X \rho\nu, \quad \nu \in \Gamma(X, \Omega^n)$$

where ρ is a compactly supported smooth function on X, constant in a neighbourhood of any point s of $Z = X - U$. Let us notice that

$$<\partial\gamma, \omega> = (-1)^n \int \omega \wedge d\rho, \quad \omega \in \Gamma(U, \Omega^{n-1}), \ d\omega = 0.$$

In order to calculate $\operatorname{Ind}(\gamma; s)$ we replace U by a small pointed neighbourhood D^* of s. With the notation of (7.2) let us write $\rho = \rho(s)\rho_s$ and deduce that

$$<\partial\gamma, \omega> \; = \; \rho(s)\operatorname{Tr}(\omega; s), \quad \omega \in \Gamma(D^*, \Omega^{n-1}), \; d\omega = 0 .$$

We can now conclude from (7.6) that

$$\operatorname{Ind}(\gamma; s) = \rho(s), \quad s \in X - U .$$

This reveals that $s \mapsto \operatorname{Ind}(\gamma; s)$ is a compactly supported, locally constant function on $X - U$.

For a given fixed point $s \notin \operatorname{Supp}(b\gamma)$ choose U to be an open neighbourhood of $\operatorname{Supp}(b\gamma)$ with $\bar{U}$ compact and $s \notin U$. We can apply the considerations above and conclude that the winding number is constant in a neighbourhood of s and zero outside some compact neighbourhood of $\operatorname{Supp}(b\gamma)$. Q.E.D.

(7.11) COROLLARY. *Let* γ *be a compact n-chain on the oriented smooth manifold* X *and* U *an open subset of* X *containing* $\operatorname{Supp}(b\gamma)$. *The relative de Rham homology class*

$$[\gamma] \in H_n^c(X, U; \mathbf{C})$$

is zero if and only if $\operatorname{Ind}(\gamma; s) = 0$ *for all* $s \in X - U$.

Proof. This is a corollary to the proof of (7.10) rather than the statement (7.10). Anyway, the basic point is Poincaré duality (6.6). Q.E.D.

8. CAUCHY'S RESIDUE THEOREM

We shall consider a smooth map $\gamma: S^{n-1} \to E$ from the oriented $n-1$ sphere into an oriented n-dimensional real vector space E. For a point s outside $\gamma(S^{n-1})$ pick a closed $(n-1)$-form ω_s on $E - \{s\}$ with $\operatorname{Tr}(\omega_s; s) = 1$ and define the *winding number* of γ with respect to s to be

$$\operatorname{Ind}(\gamma; s) = \int_{S^{n-1}} \gamma^* \omega_s . \tag{8.1}$$

(8.2) CAUCHY'S RESIDUE THEOREM. *Let* $\gamma: S^{n-1} \to X$ *denote a smooth map into an open subset* X *of* E *with* $\operatorname{Ind}(\gamma; z) = 0$ *for all* $z \in E - X$.