

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 35 (1989)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON SUMS OF FOURIER COEFFICIENTS OF CUSP FORMS
Autor: Hafner, James Lee / Ivi, Aleksandar
DOI: <https://doi.org/10.5169/seals-57381>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

ON SUMS OF FOURIER COEFFICIENTS OF CUSP FORMS

by James Lee HAFNER and Aleksandar IVIĆ

ABSTRACT. In this paper, we provide both upper and lower estimates for the summatory functions of coefficients of cusp forms. Some of the results also hold for Maass wave forms. The proofs are essentially applications of general results of Chandrasekharan and Narasimhan and of Redmond together with some non-trivial results of Murty, Rankin and others.

The purpose of this note is to collect some hitherto unnoticed or unpublished results concerning the summatory functions of the Fourier coefficients of cusp forms, including Maass wave forms. Most of the results are obtained by direct application of some general theorems about Dirichlet series satisfying a functional equation and more specific (and deeper) results concerning cusp form coefficients. One of our purposes in writing this paper is to get some of the "folklore" into print.

As usual, we will need some notation. Let $F(z)$ be a cusp form of weight k for $\Gamma = PSL(2, \mathbb{Z})$. (Our results also hold for cusp forms on congruence subgroups but this extra generality would only make the notation more complicated.) Write $F(z)$ in a Fourier series:

$$F(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}. \quad (\text{Im } z > 0)$$

If we assume that F is a normalized eigenfunction for the Hecke operators, then we have that the $a(n)$'s are multiplicative, real valued, and satisfy

$$|a(n)| \leq d(n)n^{(k-1)/2},$$

and

$$(1) \quad \sum_{n \leq x} a(n)^2 = Ax^k + O(x^{k-2/5}).$$

Here $A > 0$ depends on F in an explicit way. The first of these is a very deep result of Deligne [5] (and was called the Ramanujan-Petersson conjecture); the second is a fundamental result of Rankin [14]. Also, the Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

is absolutely convergent if $\operatorname{Re} s > (k+1)/2$ and satisfies the functional equation

$$\xi(s) = (2\pi)^{-s} \Gamma(s) L(s) = (-1)^{k/2} \xi(k-s).$$

The most famous example of the Fourier coefficients of a cusp form is the Ramanujan-tau function defined by the identity

$$\Delta(z) = \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}. \quad (\operatorname{Im} z > 0)$$

The function $\Delta(z)$ is a cusp form for Γ of weight $k = 12$. See Apostol [1] for this example and some of the basic results mentioned above.

In this paper we study the summatory function

$$A(x) = \sum_{n \leq x} a(n)$$

and the error term in the formula

$$\int_0^x A^2(t) dt = Cx^{k+1/2} + B(x),$$

where $C > 0$ is given explicitly by

$$C = \frac{1}{(4k+2)\pi^2} \sum_{n=1}^{\infty} \frac{a^2(n)}{n^{k+1/2}}.$$

It is known that

$$B(x) = \begin{cases} O(x^k \log^2 x), \\ \Omega\left(x^{k-1/4} \frac{(\log \log \log x)^3}{\log x}\right). \end{cases}$$

The upper bound was obtained by Walfisz [21] (in the special case of the Ramanujan τ -function, but the general case follows the same lines), and was reproved by Chandrasekharan and Narasimhan [3] in greater generality. The omega result may be found in Ivić [6, Theorem 4].

Our first result is a slight improvement over what has probably been in the folklore for years, but does not seem to have appeared in print before.

THEOREM 1. *We have*

$$A(x) = O(x^{(k-1)/2 + 1/3}).$$

Proof. First, by partial summation and a result found in Perelli [13], we observe that

$$\sum_{p \leq x} \frac{|a(p)|^2}{p^k} = \log \log x + O(1).$$

This implies by the Cauchy-Schwarz inequality and the well-known estimate $\sum_{p \leq x} 1/p = \log \log x + O(1)$ that

$$\begin{aligned} \sum_{p \leq x} |a(p)| p^{(1-k)/2} p^{-1} &\leq \left\{ \sum_{p \leq x} \frac{|a(p)|^2}{p^k} \right\}^{1/2} \left\{ \sum_{p \leq x} \frac{1}{p} \right\}^{1/2} \\ &= \log \log x + O(1). \end{aligned}$$

Hence by Deligne's estimate and a result of Shiu [20], we have uniformly for $x^\varepsilon \ll y \leq x$,

$$\begin{aligned} \sum_{x < n \leq x+y} |a(n)| &\ll x^{(k-1)/2} \sum_{x < n \leq x+y} |a(n)| n^{(1-k)/2} \\ &\ll \frac{x^{(k-1)/2} y}{\log x} \exp \left\{ \sum_{p \leq x+y} |a(p)| p^{(1-k)/2} p^{-1} \right\} \\ (2) \quad &\ll x^{(k-1)/2} y. \end{aligned}$$

Now all we need to do to complete the proof of the theorem is to apply Chandrasekharan and Narasimhan's general Theorem 4.1 and Remark (5.5) of [2] with $\delta = k$, $A = 1$, $q = -\infty$, $u = 1/4$ and $\eta = 1/6$ to deduce the theorem. We have taken $y = x^{1/2-\eta}$. In the folklore, the result is given with an additional factor of a power of $\log x$ which arises by using a weaker form of Shiu's result. (The first appearance in print of a result of this kind, without proof and with x^ε instead of $\log x$, can be found in [15].)

In correspondence, Rankin has claimed a similar result with a negative power of the logarithm:

$$A(x) = O(x^{(k-1)/2 + 1/3} (\log x)^{-\delta + \varepsilon}),$$

where $\delta = (8 - 3\sqrt{6})/10$. This uses much deeper and more precise information

about the distribution of values of $|a(n)|$, including the analytic behavior of the Dirichlet series $\sum_{n=1}^{\infty} a(n)^4 n^{-s}$ due to Moreno and Shahidi [10].

Our next result is a significant improvement over the results of Joris [8] and Redmond [18]. It also gives the natural analogue of the Ω_- results of Corrádi and Kátai [4] for Dirichlet's divisor problem.

THEOREM 2. *There exists a positive constant D such that*

$$A(x) = \Omega_{\pm} \left(x^{k/2-1/4} \exp \left\{ D \frac{(\log \log x)^{1/4}}{(\log \log \log x)^{3/4}} \right\} \right).$$

Proof. This result is an immediate consequence of the general theorem of Redmond [18] (actually, the Corollary to Theorem 2) and the following result of Murty [11]: for any $\varepsilon > 0$, the inequality

$$|a(p)| > (\sqrt{2} - \varepsilon) p^{(k-1)/2}$$

holds for a positive proportion of primes p . To apply Redmond's result, we let P be this set of primes (with $\varepsilon = .1$, say). Then the hypotheses of the cited Corollary are satisfied and the result follows.

The key to this method is a lower bound for the sum

$$\sum_{q \in Q_x} |a(q)| q^{-k/2-1/4},$$

where Q_x is the set of all square-free numbers composed entirely of primes less than or equal to x and in P . Joris obtains only the lower bound $c \log x$. In our first success with this problem, we used the simple inequality

$$\sum_{q \in Q_x} |a(q)| q^{-k/2-1/4} \gg \sum_{n \leq x} |a(n)| n^{-k/2-1/4}$$

and the deeper result of Rankin [17] that the latter sum is bounded below by $cx^{1/4}(\log x)^{2-1/2-1}$. The sum in question however can be written as a finite Euler product, in which case Murty's result gives the much improved lower bound

$$c' \exp \{ cx^{1/4}/\log x \}.$$

Finally, we give an improved omega result for $B(x)$.

THEOREM 3. *With D given by Theorem 2,*

$$B(x) = \Omega \left(x^{k-1/4} \exp \left\{ 3D \frac{(\log \log x)^{1/4}}{(\log \log \log x)^{3/4}} \right\} \right).$$

Proof. The method of proof is exactly that used in Ivić-Ouellet [7] where it is applied to the analogous problem for the divisor function. It is also the same method used by Ivić [6] to obtain the Ω -result mentioned in (3) above. In that paper however, he did not have the stronger omega result of Theorem 2. We reconstruct the proof here for completeness.

First note that, uniformly for $x^\varepsilon \ll H \leq x$,

$$A(x) = H^{-1} \int_x^{x+H} A(t) dt + O(x^{(k-1)/2} H).$$

This can easily be shown using Deligne's and Shiu's bounds already applied above in (2).

For simplicity of notation, put

$$T(x) = \exp \left\{ D \frac{(\log \log x)^{1/4}}{(\log \log \log x)^{3/4}} \right\}.$$

Note that $T(x)$ is increasing for x sufficiently large and that $T(2x)/T(x) \rightarrow 1$ as x tends to infinity.

Suppose the theorem does not hold so that for any $\varepsilon > 0$ and for $x > x_0(\varepsilon)$,

$$B(x) \leq \varepsilon^2 x^{k-1/4} T(x)^3.$$

We may assume that $x_0(\varepsilon)$ is large enough so that $T(x)$ is increasing for $x > x_0(\varepsilon)$.

Now let x be a generic point where the Ω_+ -result of Theorem 2 holds. Then by the Cauchy-Schwarz inequality, we have, for $x^\varepsilon \ll H \leq x$ and some positive constants c_1 , c_2 , and c_3 ,

$$\begin{aligned} c_1 x^{k/2-1/4} T(x) &\leq H^{-1/2} \left(\int_x^{x+H} A^2(t) dt \right)^{1/2} + c_2 H x^{(k-1)/2} \\ &\leq c_3 x^{k/2-1/4} + H^{-1/2} |B(x+H) - B(x)|^{1/2} + c_2 H x^{(k-1)/2} \\ &\leq c_3 x^{k/2-1/4} + 2\varepsilon H^{-1/2} x^{k/2-1/8} T(2x)^{3/2} + c_2 H x^{(k-1)/2}. \end{aligned}$$

We put $H = \varepsilon x^{1/4} T(2x)$, divide this last relation by $x^{k/2-1/4} T(x)$ and let x tend to infinity (through the sequence of special values where the Ω_+ -result of Theorem 2 holds). In so doing we deduce

$$c_1 \leq 2\varepsilon^{1/2} + c_2 \varepsilon.$$

Since, by assumption, ε could be arbitrarily small, this contradiction proves the theorem.

It seems reasonable to conjecture that

$$B(x) \ll x^{k-1/4+\varepsilon}.$$

If true, this conjecture is very strong since it implies (by the method of proof given above) the classical conjecture

$$A(x) \ll x^{k/2-1/4+\varepsilon}.$$

In conclusion we make a few remarks on extending these results to Maass wave forms. For more background on Maass wave forms and their role in analytic number theory, see the article by Iwaniec in [16]. A Maass wave form $f(z)$ is an automorphic function for Γ , i.e., a function of weight zero which is an eigenfunction for the hyperbolic Laplacian. It has a Fourier series expansion (in $x = \operatorname{Re} z$) of the form

$$f(z) = \sum_{n=1}^{\infty} a_n \sqrt{y} K_{ir}(2\pi ny) \begin{cases} \cos(2\pi nx) & \text{if } f \text{ is even} \\ \sin(2\pi nx) & \text{if } f \text{ is odd,} \end{cases}$$

where K_{ir} the K -Bessel function and r is related to f 's Laplacian eigenvalue. Note that these forms are not holomorphic, only real-analytic.

One usually assumes that f is also an eigenfunction for the Hecke operators (as we did above for holomorphic forms) so that, properly normalized, the coefficients a_n are multiplicative and real. The associated Dirichlet series satisfies a (slightly more complicated) functional equation (see Maass [9]) so the theorems of Chandrasekharan-Narasimhan [2] and Redmond [18] still apply. Rankin's result (1) carries over to this case easily. (One takes $k = 1$ in that result and elsewhere because of a slightly odd normalization.) However, the Ramanujan-Petersson conjecture $|a_n| \leq d(n)$ has not been settled for these coefficients. The most that has been shown is

$$(3) \quad |a_n| \ll n^{1/5+\varepsilon}$$

and

$$|a_p| > 1.189$$

for a positive proportion of primes p . The first was observed independently by a number of people and follows from the work of Shahidi [19]. The second is due to Murty. See Murty's paper [12] for a proof of both.

Consequently, Theorem 2 above holds for these coefficients *mutatis mutandis* but Theorems 1 and 3 do not as they depend on the inequality (2) which requires the Ramanujan-Petersson conjecture. Using Rankin's result (1) or inequality (3), one can of course give a weaker estimate for the sum

$$\sum_{x < n \leq x+y} |a_n|.$$

However, this would not be sufficient to prove an analogue of Theorem 3 (by the methods of this paper) but does yield the bound, analogous to Theorem 1,

$$\sum_{n \leq x} a_n \ll x^{2/5}.$$

REFERENCES

- [1] APOSTOL, T. *Modular Functions and Dirichlet Series in Number Theory*. GTM 41, Springer-Verlag, Berlin-Heidelberg-New York, 1976.
- [2] CHANDRASEKHARAN, K. and R. NARASIMHAN. Functional equations with multiple gamma factors and the average order of arithmetical functions. *Ann. of Math.* 76 (1962), 93-136.
- [3] CHANDRASEKHARAN, K. and R. NARASIMHAN. On the mean value of the error terms for a class of arithmetical functions. *Acta Math.* 112 (1964), 41-67.
- [4] CORRÁDI, K. and I. KÁTAI. Egy megjegyzés K. S. Gangadharan, "Two classical lattice point problems" címu dolgozatához. *MTA III Ostály Közleményei* 17 (1967), 89-97.
- [5] DELIGNE, P. La conjecture de Weil. I. *Inst. Hautes Etudes Sci. Publ. Math.* 43 (1974), 273-307.
- [6] IVIĆ, A. Large values of certain number-theoretic error terms. Preprint.
- [7] IVIĆ, A. and M. OUELLET. Some new estimates in the Dirichlet divisor problem. *Acta Arith.* 52 (1989), 39-51.
- [8] JORIS, H. Ω -Sätze für zwei arithmetische Funktionen. *Comment. Math. Helv.* 47 (1972), 220-248.
- [9] MAASS, H. Über eine neue Art von nicht analytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen. *Math. Ann.* 121 (1949), 141-183.
- [10] MORENO, C. and F. SHAHIDI. The fourth moment of the Ramanujan τ -function. *Math. Ann.* 266 (1983), 233-239.
- [11] MURTY, M. R. Oscillations of Fourier coefficients of modular forms. *Math. Ann.* 262 (1983), 431-446.
- [12] — On the estimation of eigenvalues of Hecke operators. *Rocky Mount. J.* 15 (1985), 521-533.
- [13] PERELLI, A. On the prime number theorem for the coefficients of certain modular forms. In *Elementary and analytic theory of numbers*, Banach Center Pubs. 17, PWN-Polish Scientific Publishers, Warsaw 1985, 405-410.
- [14] RANKIN, R. A. Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions. II, The order of Fourier coefficients of integral modular forms. *Math. Proc. Cambridge Phil. Soc.* 35 (1939), 357-372.
- [15] — Ramanujan's function $\tau(n)$. In *Symposia on Theoretical Physics and Mathematics*, Vol. 10 (Inst. Math. Sci. Madras, 1969), 37-45, Plenum, New York, 1970.

- [16] ——— *Modular Forms*. Ellis Horwood Ltd., Chichester, England, 1984.
- [17] ——— Sums of powers of cusp form coefficients, II. *Math. Ann.* 272 (1985), 593-600.
- [18] REDMOND, D. Omega theorems for a class of Dirichlet series. *Rocky Mount. J.* 9 (1979), 733-748.
- [19] SHAHIDI, F. On certain L -functions. *Amer. J. Math.* 103 (1981), 297-355.
- [20] SHIU, P. A Brun-Titchmarsh theorem for multiplicative functions. *J. Reine Angew. Math.* 31 (1980), 161-170.
- [21] WALFISZ, A. Über die Koeffizientensummen einiger Modulformen. *Math. Ann.* 108 (1933), 75-90.

(Reçu le 20 avril 1989)

James Lee Hafner

IBM Research Division K53/802
Almaden Research Center
650 Harry Road
San Jose, CA 95120-6099
USA

Aleksandar Ivić

Katedra Matematike RGF-a
Universiteta u Beogradu
Djusina 7
11000 Beograd
Jugoslavija

vide-leer-empty

vide-leer-empty