Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 35 (1989)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON CHERN CLASSES OF FINITE GROUP REPRESENTATIONS
Autor: Kroll, Ove
Kapitel: Section 3. Proof of Theorem 5
DOI: https://doi.org/10.5169/seals-57380

$$\varphi: T_n(q) \to \mu_\infty ,$$
$$c_\bullet^{(1)}\big(d_{p_1}(\varphi)\big) = c_\bullet^{(2)}\big(d_{p_2}(\varphi)\big) .$$

But

$$d_{p_i}(\varphi) = e_{p_i}^{-1} \circ \varphi$$

and

$$c_\bullet^{(i)}\big(d_{p_i}(\varphi)\big) = \varphi^* \circ (e_{p_i}^{-1})^* \big(e_{p_i}^*(u)\big) = \varphi^*(u) .$$

Remark. It is necessary to reduce to $\mathbf{Z}_{l^a}$ coefficients as the restriction map

$$H^*(Gl_n(q), \hat{\mathbf{Z}}_l) \to H^*(T_n(q), \hat{\mathbf{Z}}_l)$$

is not injective in general.

Section 3. Proof of Theorem 5

CH1 and CH2 clearly follow from resp. CH1 and CH2 in Theorem 2 together with the functoriality of the decomposition map d_p i.e. the diagram

$$\begin{array}{ccc} R_{\mathbf{C}}(G) & \xrightarrow{f^*} & R_{\mathbf{C}}(H) \\ \downarrow d_p & & \downarrow d_p \\ R_p(G) & \xrightarrow{f^*} & R_p(H) \end{array}$$

is commutative for a group homomorphism $f: H \to G$. To obtain CH3 note that $d_p(\varphi) = e_p^{-1} \circ \varphi$ so by definition

$$c_1(\varphi) = c_1\big(d_p(\varphi)\big) = (e_p^{-1} \circ \varphi)(e_p^* u) = \varphi^* \circ (e_p^{-1})^* \circ e_p^*(u) = \varphi^*(u) .$$

Furthermore let δ be the connecting homomorphism obtained from the exact sequence

$$\mathbf{Z} \rightarrowtail \mathbf{Q} \twoheadrightarrow \mathbf{Q}/\mathbf{Z}$$

As the diagram

$$\begin{array}{ccc} H^1(\mu_\infty, \mathbf{Q}/\mathbf{Z}) & \overset{\sim\delta}{\to} & H^2(\mu_\infty, \mathbf{Z}) \\ \downarrow \varphi^* & & \downarrow \varphi^* \\ H^1(G, \mathbf{Q}/\mathbf{Z}) & \overset{\sim\delta}{\to} & H^2(G, \mathbf{Z}) \end{array}$$

is commutative and as both δ's are isomorphisms it suffices to show that

$$\delta^{-1} \circ c_1 : \mathrm{Hom}\,(G, \mathbf{C}^*) \to H^1(G, \mathbf{Q}/\mathbf{Z}) \simeq \mathrm{Hom}\,(G, \mathbf{Q}/\mathbf{Z})$$

is an isomorphism.

But by inspection $\delta^{-1} \circ c_1(\varphi) = \delta^{-1}(u) \circ \varphi$, and as u is a $\hat{\mathbf{Z}} = \mathrm{End}_{\mathbf{Z}}(\mu_\infty)$ generator for $H^2(\mu_\infty, \mathbf{Z}) \simeq \hat{\mathbf{Z}}$, $\delta^{-1}(u)$ is an isomorphism

$$\delta^{-1}(u) : \mu_\infty \xrightarrow{\sim} \mathbf{Q}/\mathbf{Z}\,.$$

Section 4. Chern classes for locally finite groups

The definition will be based on the following two observations. In the following, let $G = \varinjlim G_k$ be a locally finite group where $\{G_k\}$ is a family of finite subgroups.

Lemma. *Let*

$$\varphi : G \to Gl_n(\mathbf{C})$$

be a representation of G. *Then* φ *is uniquely determined by its restrictions*

$$\varphi_k : G_k \to Gl_n(\mathbf{C})\,.$$

Conversely given a family of compatible representations $\varphi_k : G_k \to Gl_n(\mathbf{C})$, *there exists a unique* $\varphi : G \to Gl_n(\mathbf{C})$ *which restricts to* φ_k *for all* k.

Proof. From the universal property of the direct limit, we have

$$\mathrm{Hom}\,(G, Gl_n(\mathbf{C})) \cong \varprojlim \mathrm{Hom}\,(G_k, Gl_n(\mathbf{C}))\,.$$

Proposition. *For all* $i \geqslant 0$, *the natural map*

$$H^i(G, \mathbf{Z}) \cong \varprojlim H^i(G_k, \mathbf{Z})\,.$$

is an isomorphism.

Proof. Obvious for $i = 0, 1$. For $i \geqslant 1$, the homology groups

$$H_i(G, \mathbf{Z}) = \varinjlim H_i(G_k, \mathbf{Z})$$

are all abelian torsion groups.

Now by the universal coefficient theorem $(i \geqslant 1)$