

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 35 (1989)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON CHERN CLASSES OF FINITE GROUP REPRESENTATIONS  
**Autor:** Kroll, Ove  
**Kapitel:** Section 2. Proof of Theorem 4  
**DOI:** <https://doi.org/10.5169/seals-57380>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 28.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

To prove CH3, observe that for  $G$  locally finite the homology groups  $H_i(G, \mathbf{Z})$  are all torsion groups for  $i > 0$  as

$$H_i(G, \mathbf{Z}) \cong \varinjlim H_i(G_k, \mathbf{Z}),$$

the limit taken over a family of finite subgroups  $G_k$  of  $G$  such that  $\varinjlim G_k = G$ . Now, by the universal coefficient theorem,

$$0 \rightarrow \text{Ext}_{\mathbf{Z}}^1(H_1(G, \mathbf{Z}), \hat{\mathbf{Q}}_l) \rightarrow H^2(G, \hat{\mathbf{Q}}_l) \rightarrow \text{Hom}_{\mathbf{Z}}(H_2(G, \mathbf{Z}), \hat{\mathbf{Q}}_l) \rightarrow 0$$

is exact ( $\hat{\mathbf{Q}}_l$  is the quotient field of  $\hat{\mathbf{Z}}_l$ ) so it follows that  $H^2(G, \hat{\mathbf{Q}}_l) = 0$  as  $\hat{\mathbf{Q}}_l$  is both torsion-free and divisible. From the long exact sequence in cohomology it now follows that

$$H^1(G, \hat{\mathbf{Q}}_l/\hat{\mathbf{Z}}_l) \cong H^2(G, \hat{\mathbf{Z}}_l).$$

Finally, as  $\hat{\mathbf{Q}}_l/\mathbf{Z}_l \cong C_{l^\infty}$ , where  $C_{l^\infty}$  is the injective hull of a cyclic  $l$ -group, it follows that

$$\prod_{l \neq p} H^2(G, \hat{\mathbf{Z}}_l) \cong \prod_{l \neq p} H^1(G, C_{l^\infty}) \cong H^1(G, \prod_{l \neq p} C_{l^\infty}) = H^1(G, \bigoplus_{l \neq p} C_{l^\infty}).$$

The last equality holds, as  $G$  is locally finite and  $\bigoplus_{l \neq p} C_{l^\infty}$  is the torsion subgroup of  $\prod_{l \neq p} C_{l^\infty}$ .

SECTION 2. PROOF OF THEOREM 4

Let  $G$  be a given finite group of order  $|G|$  and

$$\rho: G \rightarrow Gl_n(\mathbf{C})$$

a complex representation.

Choose  $q$  to be a power of a prime number  $p$  different from  $l$  such that

$$q \equiv 1 \pmod{|G|}$$

Define

$$\phi: Gl_n(q) \rightarrow \mathbf{C}$$

by

$$\phi(g) = \sum_{i=1}^n e_p(\lambda_i)$$

where  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of  $g$ . As shown by J. A. Green in [4],  $\phi$  is a virtual complex character of  $Gl_n(q)$ .

Furthermore let

$$f: G \rightarrow Gl_n(q)$$

be the mod- $p$  reduction of  $\rho$  to  $Gl_n(q)$ . (It factors through  $Gl_n(q)$ , as all  $|G|$ -roots of unity are contained in the Galois field  $GF(q)$  with  $q$  elements).

Let  $f^*: R_{\mathbf{C}}(Gl_n(q)) \rightarrow R_{\mathbf{C}}(G)$  be a map induced on complex character rings by  $f$ . By inspection

$$f^*(\phi) = \rho.$$

Let  $a = v_l(q-1)$ , where  $v_l$  is the  $l$ -adic valuation and let

$$p^{**}: H^{**}(G, \hat{\mathbf{Z}}_l) \rightarrow H^{**}(G, \mathbf{Z}_{l^a})$$

be the map induced by the projection  $p: \hat{\mathbf{Z}}_l \rightarrow \mathbf{Z}_{l^a}$ . Clearly  $p^{**}$  is injective in positive dimensions, as multiplication by  $l^a$  is zero on  $H^{**}(G, \hat{\mathbf{Z}}_l)$ .

Now the following diagram is commutative

$$\begin{array}{ccc} H^{**}(G, \hat{\mathbf{Z}}_l) & \xrightarrow{p^{**}} & H^{**}(G, \mathbf{Z}_{l^a}) \\ f^{**} \uparrow & & \uparrow f^{**} \\ H^{**}(Gl_n(q), \hat{\mathbf{Z}}_l) & \rightarrow & H^{**}(Gl_n(q), \mathbf{Z}_{l^a}) \\ \downarrow \text{res} & & \downarrow \text{res} \\ H^{**}(T_n(q), \hat{\mathbf{Z}}_l) & \xrightarrow{p^{**}} & H^{**}(T_n(q), \mathbf{Z}_{l^a}) \end{array}$$

where the restriction map on the right is injective as shown in [6].

Thus for  $i = 1, 2$

$$c^{(i)}(d_{p_i}(\rho)) = (p^{**})^{-1} f^{**}(\text{res})^{-1} p^{**}(d_{p_i}(\tau))$$

where  $\tau$  is the restriction of the virtual character  $\phi$  to  $T_n(q)$ . [Note that  $(p^{**})^{-1}$  and  $(\text{res})^{-1}$  both make sense as the above diagram is commutative].

Thus to show equality, it suffices using CH1 in Theorem 2, to show that

$$c^{(1)}(d_{p_1}(\tau)) = c^{(2)}(d_{p_2}(\tau))$$

But  $T_n(q)$  is abelian so  $\tau$  is a direct sum of  $n$  one-dimensional representations. By CH2 of Theorem 2 it suffices to show that for a one dimensional representation

$$\begin{aligned} \varphi: T_n(q) &\rightarrow \mu_\infty, \\ c^{(1)}(d_{p_1}(\varphi)) &= c^{(2)}(d_{p_2}(\varphi)). \end{aligned}$$

But

$$d_{p_i}(\varphi) = e_{p_i}^{-1} \circ \varphi$$

and

$$c^{(i)}(d_{p_i}(\varphi)) = \varphi^* \circ (e_{p_i}^{-1})^* (e_{p_i}^*(u)) = \varphi^*(u).$$

*Remark.* It is necessary to reduce to  $\mathbf{Z}_{l^a}$  coefficients as the restriction map

$$H^*(Gl_n(q), \hat{\mathbf{Z}}_l) \rightarrow H^*(T_n(q), \hat{\mathbf{Z}}_l)$$

is not injective in general.

### SECTION 3. PROOF OF THEOREM 5

CH1 and CH2 clearly follow from resp. CH1 and CH2 in Theorem 2 together with the functoriality of the decomposition map  $d_p$  i.e. the diagram

$$\begin{array}{ccc} R_c(G) & \xrightarrow{f^*} & R_c(H) \\ \downarrow d_p & & \downarrow d_p \\ R_p(G) & \xrightarrow{f^*} & R_p(H) \end{array}$$

is commutative for a group homomorphism  $f: H \rightarrow G$ . To obtain CH3 note that  $d_p(\varphi) = e_p^{-1} \circ \varphi$  so by definition

$$c_1(\varphi) = c_1(d_p(\varphi)) = (e_p^{-1} \circ \varphi)^*(e_p^*(u)) = \varphi^* \circ (e_p^{-1})^* \circ e_p^*(u) = \varphi^*(u).$$

Furthermore let  $\delta$  be the connecting homomorphism obtained from the exact sequence

$$\mathbf{Z} \rightarrow \mathbf{Q} \rightarrow \mathbf{Q}/\mathbf{Z}$$

As the diagram

$$\begin{array}{ccc} H^1(\mu_\infty, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\sim \delta} & H^2(\mu_\infty, \mathbf{Z}) \\ \downarrow \varphi^* & & \downarrow \varphi^* \\ H^1(G, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\sim \delta} & H^2(G, \mathbf{Z}) \end{array}$$