

Section 1. Proof of Theorem 2

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **35 (1989)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **27.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

$$c(\rho \circ f) = f^*(c \cdot (\rho)).$$

$$CH2. \quad c \cdot (\rho_1 \oplus \rho_2) = c \cdot (\rho_1) \cdot c \cdot (\rho_2).$$

CH3. $c_1: \text{Hom}(G, \mathbf{C}^*) \rightarrow H^2(G, \mathbf{Z})$ is an isomorphism and can be described as follows: For $\varphi \in \text{Hom}(G, \mathbf{C}^*)$, let φ also denote its unique factorization

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & C^* \\ & \searrow \varphi & \uparrow \cup \\ & & \mu_\infty \end{array}$$

$$\text{Now } c_1(\varphi) = \varphi^*(u).$$

Remark. As shown in [7], CH1, CH2 and CH3 uniquely determine the Chern classes defined by u . As different choices of u clearly defines different Chern classes (just observe that

$$H^2(\mu_\infty, \mathbf{Z}) \cong \lim_{\rightarrow} H^2(G_i, \mathbf{Z}),$$

the limit taken over all finite cyclic subgroups), there is a one-to-one correspondence between Chern classes on finite groups and $\hat{\mathbf{Z}}$ generators of $H^2(\mu_\infty, \mathbf{Z})$.

This paper has been organized as follows.

Theorem 2 is proved in Section 1, Theorem 4 in Section 2, and Theorem 5 in Section 3. Proposition 3 i) was proved in [7], and the remaining part of this proposition can be obtained similarly.

Finally, in Section 4 it is shown that there exists a very simple extension of the theory of Chern classes on finite groups to locally finite groups.

I would like to thank Jørgen Tornehave for a helpful conversation.

SECTION 1. PROOF OF THEOREM 2

CH1 is quite trivial, so let me first prove CH2. Let $\dim \rho_i = n_i$, $\dim \rho = n$, so that $n_1 + n_2 = n$. By assumption, ρ factors through the parabolic subgroup $P = P(k_p)$

$$P = \left\{ \begin{array}{cc} n_1 & n_2 \\ * & * \\ 0 & * \end{array} \right\}$$

which is isomorphic to a semi-direct product of $Gl_{n_1}(\bar{k}_p) \times Gl_{n_2}(\bar{k}_p)$ acting on a unipotent subgroup U .

As U is a direct limit of p -groups,

$$H^k(U, \hat{\mathbf{Z}}_l) = 0 \quad \text{for } k > 0.$$

Thus

$$\begin{aligned} H^*(P, \hat{\mathbf{Z}}_l) &\cong H^*(Gl_{n_1}(\bar{k}_p), \hat{\mathbf{Z}}_l) \otimes H^*(Gl_{n_2}(\bar{k}_p), \hat{\mathbf{Z}}_l) \\ &\cong P(\alpha_1, \dots, \alpha_{n_1}) \otimes P(\beta_1, \dots, \beta_{n_2}). \end{aligned}$$

Let

$$H^*(Gl_n(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(\sigma_1, \dots, \sigma_n)$$

and

$$H^*(T_n(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(x_1, \dots, x_n).$$

As $T_n(\bar{k}_p) \cong T_{n_1}(\bar{k}_p) \times T_{n_2}(\bar{k}_p)$, I shall consider

$$H^*(T_{n_1}(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(x_1, \dots, x_{n_1})$$

and

$$H^*(T_{n_2}(\bar{k}_p), \hat{\mathbf{Z}}_l) \cong P(x_{n_1+1}, \dots, x_n)$$

as contained in $H^*(T_n(\bar{k}_p), \hat{\mathbf{Z}}_l)$. Furthermore, as all restriction maps are injective, I shall view $H^*(Gl_n(\bar{k}_p), \hat{\mathbf{Z}}_l)$ and $H^*(Gl_{n_i}(\bar{k}_p), \hat{\mathbf{Z}}_l)$, $i = 1, 2$, as subspaces of $H^*(T_n(\bar{k}_p), \hat{\mathbf{Z}}_l)$. Thus

α_i = the i 'th elementary symmetric polynomial in x_1, \dots, x_{n_1}

β_i = the i 'th elementary symmetric polynomial in x_{n_1+1}, \dots, x_n

σ_i = the i 'th elementary symmetric polynomial in x_1, \dots, x_n .

Furthermore, the formula

$$c \cdot (\rho_1 \oplus \rho_2) = c \cdot (\rho_1 \oplus \rho_2)$$

is equivalent to

$$1 + \sigma_1 t + \dots + \sigma_n t^n = (1 + \alpha_1 t + \dots + \alpha_{n_1} t^{n_1}) \cdot (1 + \beta_1 t + \dots + \beta_{n_2} t^{n_2}),$$

and this follows from the identity

$$\begin{aligned} \sum_{i=0}^n \sigma_i t^i &= \prod_{i=1}^n (1 + tx_i) = \prod_{i=1}^{n_1} (1 + tx_i) \cdot \prod_{i=n_1+1}^{n_2} (1 + tx_i) \\ &= \left(\sum_{i=0}^{n_1} \alpha_i t^i \right) \left(\sum_{i=0}^{n_2} \beta_i \cdot t^i \right). \end{aligned}$$

To prove CH3, observe that for G locally finite the homology groups $H_i(G, \mathbf{Z})$ are all torsion groups for $i > 0$ as

$$H_i(G, \mathbf{Z}) \cong \lim_{\rightarrow} H_i(G_k, \mathbf{Z}),$$

the limit taken over a family of finite subgroups G_k of G such that $\varinjlim G_k = G$. Now, by the universal coefficient theorem,

$$0 \rightarrow \text{Ext}_{\mathbf{Z}}^1(H_1(G, \mathbf{Z}), \hat{\mathbf{Q}}_l) \rightarrow H^2(G, \hat{\mathbf{Q}}_l) \rightarrow \text{Hom}_{\mathbf{Z}}(H_2(G, \mathbf{Z}), \hat{\mathbf{Q}}_l) \rightarrow 0$$

is exact ($\hat{\mathbf{Q}}_l$ is the quotient field of $\hat{\mathbf{Z}}_l$) so it follows that $H^2(G, \hat{\mathbf{Q}}_l) = 0$ as $\hat{\mathbf{Q}}_l$ is both torsion-free and divisible. From the long exact sequence in cohomology it now follows that

$$H^1(G, \hat{\mathbf{Q}}_l/\hat{\mathbf{Z}}_l) \cong H^2(G, \hat{\mathbf{Z}}_l).$$

Finally, as $\hat{\mathbf{Q}}_l/\mathbf{Z}_l \cong C_{l^\infty}$, where C_{l^∞} is the injective hull of a cyclic l -group, it follows that

$$\prod_{l \neq p} H^2(G, \hat{\mathbf{Z}}_l) \cong \prod_{l \neq p} H^1(G, C_{l^\infty}) \cong H^1(G, \prod_{l \neq p} C_{l^\infty}) = H^1(G, \bigoplus_{l \neq p} C_{l^\infty}).$$

The last equality holds, as G is locally finite and $\bigoplus_{l \neq p} C_{l^\infty}$ is the torsion subgroup of $\prod_{l \neq p} C_{l^\infty}$.

SECTION 2. PROOF OF THEOREM 4

Let G be a given finite group of order $|G|$ and

$$\rho: G \rightarrow Gl_n(\mathbf{C})$$

a complex representation.

Choose q to be a power of a prime number p different from l such that

$$q \equiv 1 \pmod{|G|}$$

Define

$$\phi: Gl_n(q) \rightarrow \mathbf{C}$$

by

$$\phi(g) = \sum_{i=1}^n e_p(\lambda_i)$$