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COMPOSITION PRODUCTS AND MODELS FOR THE HOMFLY POLYNOMIAL

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ABSTRACT: We define a composition product for homfly polynomials of oriented links and we show how this operation can be used to construct in a natural way a sequence of state models due to Jones. We also present a refinement of this result in the case of closed braids. This leads us first to a new state model for the Alexander-Conway polynomial which can be interpreted as an ice-type model. Then we express the homfly polynomial of a braid diagram in terms of the Alexander-Conway polynomials of its subdiagrams. As a consequence, we obtain simple direct proofs of inequalities due to Morton, Franks and Williams. Finally we give a state model for the homfly polynomial of a closed braid.

RÉSUMÉ: Nous définissons un produit de composition pour les polynômes homfly des entrelacs orientés et nous montrons comment on peut utiliser cette opération pour construire de façon naturelle une suite de « modèles d'états » due à Jones. Nous présentons également un raffinement de ce résultat dans le cas des tresses fermées. Ceci nous conduit d'abord à un nouveau modèle d'états pour le polynôme d'Alexander-Conway qui peut s'interpréter comme un modèle « de type glace ». Puis nous exprimons le polynôme homfly d'un diagramme de tresse en termes des polynômes d'Alexander-Conway de ses sous-diagrammes. Comme conséquence, nous obtenons des preuves simples et directes d'inégalités dues à Morton, Franks et Williams. Enfin nous donnons un modèle d'états pour le polynôme homfly d'une tresse fermée.

1. INTRODUCTION

Since its discovery [1], the Alexander polynomial has played an important role in the development of knot theory. Its topological and algebraic aspects (relations with the fundamental group and the infinite cyclic cover of the complement) have been extensively studied. Its combinatorial aspects, already present in the original paper by Alexander, have recently received new attention. The starting point was Conway's work [3] which showed how a suitable normalization of the Alexander polynomial of an oriented link (which we shall call the Alexander-Conway polynomial) can be computed recursively on an arbitrary regular projection — or "diagram" — of the link by using a linear equation satisfied by the values of the polynomial on three links which are "skein related" (that is, they are represented by diagrams D^+, D^-, D° which differ only inside a small disk where they behave as depicted on Figure 3). In fact the Alexander-Conway polynomial can be described in purely combinatorial terms, as shown by Kauffman in [16]: to each diagram is associated in a simple way a polynomial in one variable which is shown to satisfy Conway's "skein equation" and to be invariant under Reidemeister moves. The polynomial is defined as a summation over a set of possible "states" of the diagram and can be viewed as the partition function of a certain model (we shall call it a "state model") in the sense of Statistical Mechanics [2].

More recently Jones, in relation with his work on Von Neumann algebras, discovered another one-variable polynomial link invariant [11, 12] which like the Alexander-Conway polynomial satisfies a skein equation. Both invariants were soon generalized by different authors [5, 26] into a two-variable polynomial which became known as the Jones-Conway or "homfly" (from the initials of the authors of [5]) polynomial. The homfly polynomial (as we have arbitrarily chosen to call it) can be defined combinatorially on diagrams, in relation with the Conway-type algorithm which allows its computation. It can also be defined via representations of Artin's braid groups in Hecke algebras, using theorems of Alexander (which asserts that every oriented link can be represented as a closed braid) and Markov (which characterizes the isotopy of closed braids). In both cases the proof of the existence of the homfly polynomial is quite sophisticated.

Kauffman [17] obtained an elegant and simple state model for the Jones polynomial which has lead to the solution of old conjectures on alternating links [17, 24, 27]. Jones ([14], see also [20] and [28]) also obtained state models for an infinite sequence of one-variable specializations of the homfly polynomial. For a rather special kind of diagram the homfly polynomial is equivalent to the Tutte polynomial of an associated plane graph [10], which has simple and well-known state models [2, 9, 29, 30]. However no state model is known for the full homfly polynomial of an arbitrary diagram. Moreover no natural topological 3-dimensional interpretation of the homfly polynomial has been found, apart from the case of the Alexander-Conway polynomial. The purpose of this paper is to present some progress towards the solution of these problems.

In Section 2 we introduce a composition product for homfly polynomials. This product allows the combinatorial definition of the homfly polynomial of a diagram for a given pair of values of the variables in terms of the homfly polynomials of its subdiagrams for other related pairs of values of the variables (Proposition 1). We show in Proposition 2 how the sequence of state models due to Jones can be derived simply from the product operation, starting from an elementary special case of the homfly polynomial. Then, motivated by some difficulties in the application of the concept of composition product to the Alexander-Conway polynomial, in Section 3 we restrict our attention to closed braids and we introduce a specified composition product for this class of diagrams (Proposition 3). This leads us first to another version of the Jones sequence of state models (Proposition 4). Then we obtain a state model for the Alexander-Conway polynomial (Proposition 6) which can be interpreted as an ice-type model (Proposition 7). As another consequence we give an expansion of the homfly polynomial of a braid diagram in terms of the Alexander-Conway polynomials of its subdiagrams (Proposition 9). This yields simple direct proofs of some inequalities due to Morton [22] and independently Franks and Williams [4] which have been helpful in the study of the braid index. Finally we combine the previous results to obtain a state model for the homfly polynomial of a closed braid (Proposition 12). We present some perspectives for further research in Section 4.

2. The composition product of homfly polynomials

2.1. Definitions

By *diagram* we mean a regular plane projection of a tame oriented link in 3-space. We shall consider diagrams as 4-regular directed plane graphs.

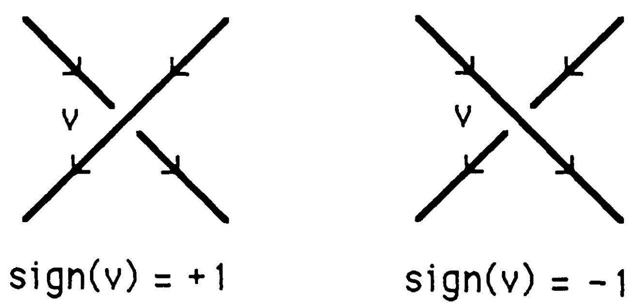

Figure 1