1. Regular states

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1. Regular states

This chapter will develop an essentially complete theory of finding minimal paths between regular states of the **TH**. It starts with an appropriate formal setting.

1.0. Mathematical model

The *pegs* will be denoted by an $i \in \{0, 1, 2\}$, the *discs* by $d \in \{1, ..., n\}$ in natural order of increasing diameter; $n \in \mathbf{N}_0$ throughout, if not otherwise stated.

Definition 0. $T_n := \{r: \{1, ..., n\} \to \{0, 1, 2\}\}$. An $r \in T_n$ will also be written as $[r(1), ..., r(n)]$.

It is evident that any regular state of the **TH** is completely described by one and only one $r \in T_n$ and that any $r \in T_n$ can be interpreted as one and only one regular state of the **TH**. So it follows immediately by induction:

THEOREM 0. *The number of regular states of the* **TH** *with* $n \in \mathbf{N}_0$ *discs is* 3^n.

Definition 1. i) A pair $(r_0, r_1) \in T_n^2$ is a *(legal) move* (of disc d from peg i to peg j), iff

$$\exists (i,j) \in \{0, 1, 2\}^2, i \neq j: \left(r_0^{-1}(\{i\}) \neq \emptyset \wedge \left(r_0^{-1}(\{j\}) = \emptyset \vee d := \min r_0^{-1}(\{i\}) < \min r_0^{-1}(\{j\})\right) \wedge \left(r_1(d) = j \wedge \forall c \in \{1, ..., n\} \setminus \{d\}: r_1(c) = r_0(c)\right)\right).$$

ii) For any pair $(s, t) \in T_n^2$ let

$$P_n(s, t) := \{p \in \bigcup_{\nu=0}^{\infty} T_n^{\nu+1}; p_0 = s, p_{\mu_p} = t \wedge \forall \mu \in \{1, ..., \mu_p\}: (p_{\mu-1}, p_\mu) \text{ is a move}\}$$

where $\mu_p := \operatorname{ind}(p)$.

A $p \in P_n(s, t)$ is called a *path* from s to t; μ_p is the *length* of p.

With this adequate formal model, it is now possible to treat $\mathfrak{P}$s 0 to 2, namely to find shortest paths between regular states. The following notions will frequently be used:

Definition 2. i) For any $r \in T_{n+1}: \bar{r} := r \mid \{1, ..., n\}$ $(\in T_n)$.

ii) For $(i, j) \in \{0, 1, 2\}^2$:

$$i \circ j := \begin{cases} i\,, & \text{if} \quad i = j\,; \\ k \in \{0, 1, 2\} \backslash \{i, j\}\,, & \text{if} \quad i \neq j\,. \end{cases}$$

(Note that $i \circ j = (-(i+j)) \bmod 3$.)

iii) For $i \in \{0, 1, 2\} : \widehat{i^n} := [i, ..., i] \in T_n$. (These are the perfect states.)

As pointed out by Er [17], it is often convenient to regard the **TH** as a graph, the vertices of which being the regular states and in which the edges are formed by the legal moves. It will turn out that this graph is planar, simple, and connected. An example ($n=3$) is given in Figure 2:

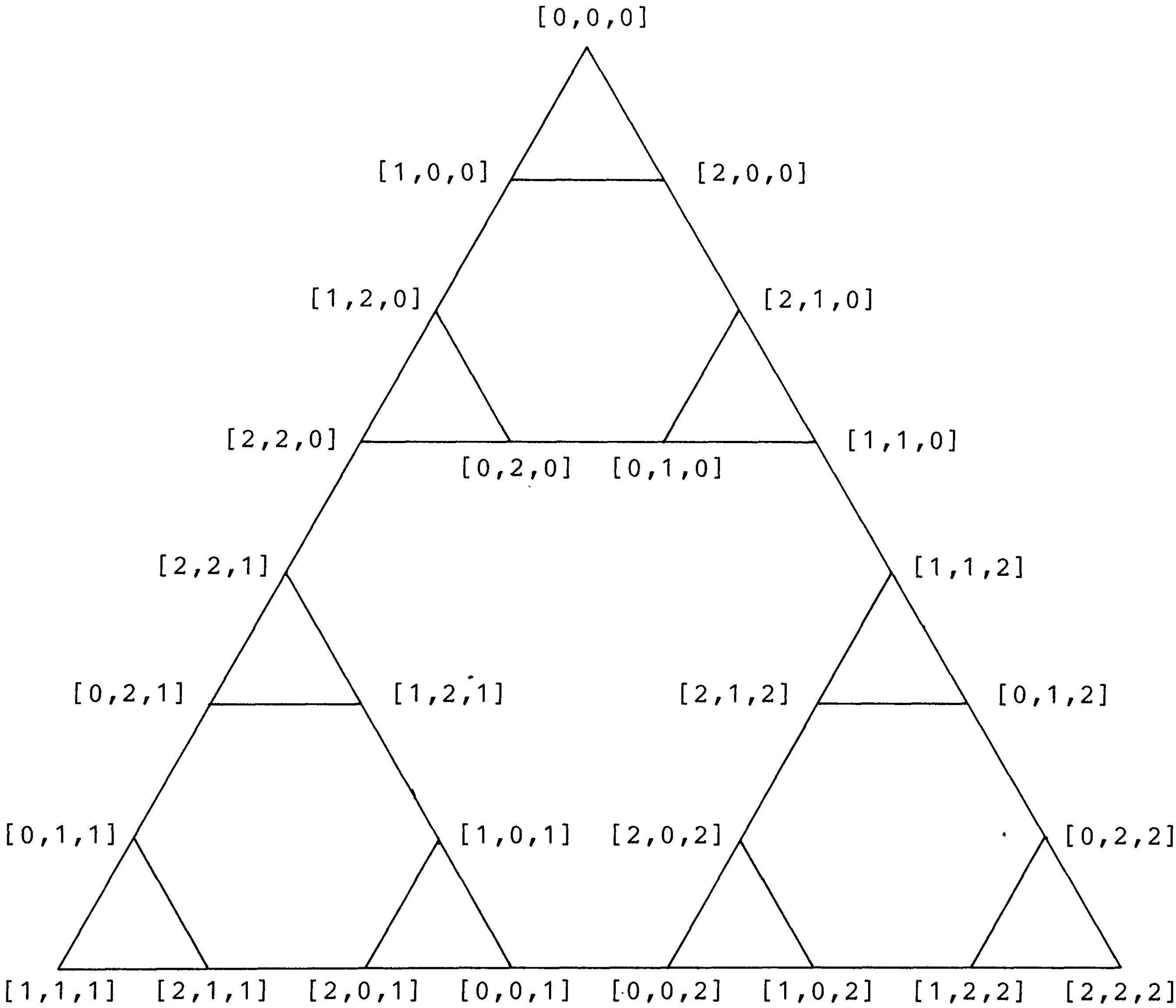

FIGURE 2.

1.1. EXISTENCE OF A SHORTEST PATH BETWEEN TWO REGULAR STATES AND AN UPPER BOUND FOR ITS LENGTH

To establish the sheer existence of a shortest path from s to t it suffices to show that $P_n(s, t) \neq \emptyset$.

THEOREM 1. *For any pair (s, t) of regular states there is a (shortest) path from s to t with length less than or equal to $2^n - 1$, where n is the number of discs involved.*

Proof by induction. a) The case $n = 0$ is trivial.

b) Let $(s, t) \in T^2_{n+1}$.

If $s(n+1) = t(n+1)$, let $\tilde{p} \in P_n(\bar{s}, \bar{t})$ with $\mu_{\tilde{p}} \leqslant 2^n - 1$, and define $p \in T^{\mu_p+1}_{n+1}$ by $\mu_p = \mu_{\tilde{p}} (\leqslant 2^{n+1} - 1)$ and $\forall \nu \in \{0, ..., \mu_p\} : \bar{p}_\nu = \tilde{p}_\nu$, $p_\nu(n+1) = s(n+1)$. It is easy to see that $p \in P_{n+1}(s, t)$.

If $s(n+1) \neq t(n+1)$, let $i := s(n+1) \circ t(n+1)$, $\tilde{p} \in P_n(\bar{s}, \hat{i})$ and $\tilde{q} \in P_n(\hat{i}, \bar{t})$ with $\mu_{\tilde{p}}, \mu_{\tilde{q}} \leqslant 2^n - 1$. Define $p \in T^{\mu_p+1}_{n+1}$ by $\mu_p = \mu_{\tilde{p}} + \mu_{\tilde{q}} + 1 (\leqslant 2^{n+1} - 1)$ and

$$\forall \nu \in \{0, ..., \mu_{\tilde{p}}\} : \bar{p}_\nu = \tilde{p}_\nu, p_\nu(n+1) = s(n+1),$$

$$\forall \nu \in \{\mu_{\tilde{p}} + 1, ..., \mu_p\} : \bar{p}_\nu = \tilde{q}_{\nu - \mu_{\tilde{p}} - 1}, p_\nu(n+1) = t(n+1).$$

Then $p \in P_{n+1}(s, t)$. □

Remark 1. The proof of Theorem 1 is constructive in that it allows to determine a path from s to t recursively.

In all papers mentioned in the introduction and dealing with $\mathfrak{P}$s 0 to 2, except those by Er [17] and Wood [49], it has been assumed that the shortest path is uniquely defined by this construction. But neither is the shortest path unique in general, nor does the construction always produce a shortest path, even if one chooses $\tilde{p}$ and $\tilde{q}$ minimal!

Example 1. a) Let $n = 2$, $s = [0, 1]$, $t = [1, 0]$,

b) Let $n = 3$, $s = [0, 0, 1]$, $t = [1, 1, 0]$.

Then a look at the graph in Figure 2 immediately shows that ([0, 1], [2, 1], [2, 0], [1, 0]) and ([0, 1], [0, 2], [1, 2], [1, 0]) are both shortest paths for a), and for b) the construction of Theorem 1 leads to the path ([0, 0, 1], [1, 0, 1], [1, 2, 1], [2, 2, 1], [2, 2, 0], [0, 2, 0], [0, 1, 0], [1, 1, 0]) of length 7, while ([0, 0, 1], [0, 0, 2], [2, 0, 2], [2, 1, 2], [1, 1, 2], [1, 1, 0]) of length 5 is shortest.

Er [17] refers to symmetry properties of the graph to establish uniqueness for $\mathfrak{P}$s 0 and 1. In [49], Wood felt the obligation to prove that the path of Theorem 1 is shortest for $s = \hat{i}$, $t = \hat{j}$ (see Section 1.2 below), but in [50], he made the mistake to assume its minimality in the case of general s and t, an error repeated by Cull and Gerety [13] (obviously, **TH** is really hard!). This problem will be treated correctly in Section 1.3.

1.2. Perfect states

This section will leave no secret about the classical $\mathfrak{P}0$. The essential step is to establish uniqueness of the shortest path between perfect states.

1.2.0. Uniqueness and length of the classical solution

THEOREM 2. *For any two distinct pegs* i *and* j, *there is exactly one shortest path from* $\hat{i}^n$ *to* $\hat{j}^n$; *its length is* $2^n - 1$.

Proof. It will be shown by induction that

$$\forall (i,j) \in \{0,1,2\}^2,\ i \neq j\ \exists_1\ p \in P_n(\hat{i},\hat{j}) : \mu_p = 2^n - 1 \text{ is minimal}.$$

a) The case $n = 0$ is trivial.

b) Let $p \in P_{n+1}(\hat{i},\hat{j})$ be shortest. As $i \neq j$, disc $n+1$ must be moved at least once. Before the first move of disc $n+1$, from i to $k \neq i$ say, discs 1 to n have to be brought from i to $i \circ k$ by the rules of a legal move of $n+1$; this is equivalent to a path from $\hat{i}^n$ to $\widehat{i \circ k}^n$, which takes at least $2^n - 1$ moves of discs 1 to n.

After the last move of $n+1$, from $l \neq j$ to j say, discs 1 to n must be brought from $l \circ j$ to j, which again takes at least $2^n - 1$ moves. So $\mu_p \geqslant 2^{n+1} - 1$.

As $\mu_p \leqslant 2^{n+1} - 1$ by Theorem 1, it follows that disc $n+1$ moves exactly once, i.e. $k = l = i \circ j$, which implies uniqueness of p too. □

Definition 3. The shortest path from $\hat{i}^n$ to $\hat{j}^n$ will be denoted by $p^{i,j;n}$.

Remark 2. Theorem 2 shows that the bound on the length of a shortest path in Theorem 1 is sharp.

1.2.1. Construction of the shortest path between two perfect states

A large part of the interest the **TH** has raised in recent years, stems from the discussion, mostly among computer scientists, which algorithm for the realization of the shortest path between perfect states is the "best". The right question is, of course: "best for what?". Four constructions will be given here, each of which suitable for a different situation. The recursive solution in o, already to be found in [8], is the backbone of the theory and fits best into textbooks on recursion. The iterative solution in i (cp. [28]), or some derived version of it, can best be used to make a computer do the **TH**. It also immediately leads to a description of the shortest path in just one formula; this algorithm ii (cp. Hering [25]) can make a parallel computer write down the solution more or less "at once". As man's mental quickness is much more limited, these algorithms are not suited to him. But there is another iterative variant iii, developped essentially in [43], allowing a human being to carry out the shortest path at a rate of about one move per second, a speed consistent with the traditional assumption of many authors.

o) Recursive algorithm. An immediate consequence of the proof of Theorem 2 is

PROPOSITION 0. *Let* $(i, j) \in \{0, 1, 2\}^2, i \neq j$. *Then*

a) $p^{i,j;0} = (\emptyset)$.

b) *For any* n, $p^{i,j;n+1}$ *is given by*

$$\forall \nu \in \{0, ..., 2^n - 1\} : \overline{p_\nu^{i,j;n+1}} = p_\nu^{i,i\circ j;n}, p_\nu(n+1) = i;$$

$$\forall \nu \in \{2^n, ..., 2^{n+1} - 1\} : \overline{p_\nu^{i,j;n+1}} = p_{\nu-2^n}^{i\circ j,j;n}, p_\nu(n+1) = j.$$

It is clear that this algorithm is of little practical interest (for large n, a huge amount of memory is needed just to do the first move!), but it serves as theoretical base for the following algorithms.

i) Iterative algorithm. This algorithm tells for the μ-th move of the shortest path which disc to move and determines its initial and final peg during that move.

Definition 4. Let $p \in P_n(s, t)$, $\mu \in \{1, ..., \mu_p\}$. Then

o) $(p_{\mu-1}, p_\mu)$ is called the μ-th move of p;

i) $d_\mu(p) :=$ disc moved in the μ-th move of p;

ii) $i_\mu(p) :=$ peg from which $d_\mu(p)$ is moved in the μ-th move of p;

iii) $j_\mu(p) :=$ peg to which $d_\mu(p)$ is moved in the μ-th move of p.

These notions are well-defined in view of Definition 1.

PROPOSITION 1. *Let* $(i, j) \in \{0, 1, 2\}^2, i \neq j$. *Then for any* $\mu \in \{1, ..., 2^n - 1\}$:

o) $d := d_\mu(p^{i,j;n}) = \min\{c \in \{1, ..., n\}; 2^c \nmid \mu\}$;

i) $i_\mu(p^{i,j;n}) = \left(\left(\frac{\mu}{2^d} - \frac{1}{2}\right)(j-i)((n-d) \bmod 2 + 1) + i\right) \bmod 3;$

ii) $j_\mu(p^{i,j;n}) = \left(\left(\frac{\mu}{2^d} + \frac{1}{2}\right)(j-i)\left((n-d) \bmod 2+1\right)+i\right) \bmod 3\,.$

Proof by induction on n.

a) For $n = 0$, the statement is trivial.

b) Proposition 0b yields: For $\mu \in \{1, ..., 2^n - 1\}$:

$$\begin{aligned} d := d_\mu(p^{i,j;n+1}) = d_\mu(p^{i,i\circ j;n}) &= \min\{c \in \{1, ..., n\};\, 2^c \nmid \mu\} \\ &= \min\{c \in \{1, ..., n+1\};\, 2^c \nmid \mu\}, \end{aligned}$$

$$\begin{aligned} i_\mu(p^{i,j;n+1}) = i_\mu(p^{i,i\circ j;n}) &= \left(\left(\frac{\mu}{2^d} - \frac{1}{2}\right)((i\circ j)-i)\left((n-d) \bmod 2+1\right)+i\right) \bmod 3 \\ &= \left(\left(\frac{\mu}{2^d} - \frac{1}{2}\right)(i-j)\left((n-d) \bmod 2+1\right)+i\right) \bmod 3 \\ &= \left(\left(\frac{\mu}{2^d} - \frac{1}{2}\right)(j-i)\left(((n+1)-d) \bmod 2+1\right)+i\right) \bmod 3\,, \end{aligned}$$

$j_\mu(p^{i,j;n+1}) = \ldots .$ (analogously);

for $\mu = 2^n$: $d = n+1$, $i_\mu(p^{i,j;n+1}) = i$, $j_\mu(p^{i,j;n+1}) = j$;
for $\mu \in \{2^n + 1, ..., 2^{n+1} - 1\}$:

$$\begin{aligned} d = d_{\mu-2^n}(p^{i\circ j,j;n}) &= \min\{c \in \{1, ..., n\};\, 2^c \nmid \mu - 2^n\} \\ &= \min\{c \in \{1, ..., n+1\};\, 2^c \nmid \mu\}, \end{aligned}$$

$$\begin{aligned} i_\mu(p^{i,j;n+1}) &= i_{\mu-2^n}(p^{i\circ j,j;n}) \\ &= \left(\left(\frac{\mu-2^n}{2^d} - \frac{1}{2}\right)(j-(i\circ j))\left((n-d) \bmod 2+1\right)+(i\circ j)\right) \bmod 3 \\ &= \left(\left(\frac{\mu}{2^d} - \frac{1}{2}\right)((i\circ j)-i)\left((n-d) \bmod 2+1\right)+i\right) \bmod 3 \\ &= \left(\left(\frac{\mu}{2^d} - \frac{1}{2}\right)(j-i)\left(((n+1)-d) \bmod 2+1\right)+i\right) \bmod 3 \end{aligned}$$

(using $\forall \kappa \in \mathbf{N}_0: 3 \mid 2^{2\kappa} - 1$),

$$j_\mu(p^{i,j;n+1}) = \ldots \text{ (analogously)}. \quad \square$$

ii) Parallel algorithm. A striking consequence of Proposition 1 is a formula which completely covers the shortest path.

PROPOSITION 2. *Let* $(i, j) \in \{0, 1, 2\}^2, i \neq j$. *Then for any* $\nu \in \{0, ..., 2^n - 1\}$ *and any* $d \in \{1, ..., n\}$:

$$p_\nu(d) := p_\nu^{i,j;n}(d) = \left((j-i)\,((n-d) \bmod 2+1)\,\mathrm{ent}\left(\frac{\nu}{2^d} + \frac{1}{2}\right) + i\right) \bmod 3 .$$

Proof. $p_0(d) = i$ and, by Proposition 1, for $\mu \in \{1, ..., 2^n - 1\}$:

$$(1) \quad p_\mu(d) = \begin{cases} p_{\mu-1}(d), & \text{if} \quad d \neq d_\mu(p); \\ (p_{\mu-1}(d) + (j-i)\,((n-d) \bmod 2+1)) \bmod 3, & \text{if} \quad d = d_\mu(p). \end{cases}$$

So $p_\nu(d) = ((j-i)\,((n-d) \bmod 2+1)\,|\{\mu \in \{1, ..., \nu\};\, d = d_\mu(p)\}| + i) \bmod 3$.
But

$$(2) \qquad d = d_\mu(p) \Leftrightarrow \exists \kappa \in \mathbf{N}_0 : \mu = 2^{d-1} + \kappa 2^d,$$

whence

$$|\{\mu \in \{1, ..., \nu\};\, d = d_\mu(p)\}| = \min\{\lambda \in \mathbf{N}_0;\, \nu < 2^{d-1} + \lambda 2^d\}$$
$$= \mathrm{ent}\left(\frac{\nu}{2^d} + \frac{1}{2}\right). \quad \square$$

The observations from Proposition 1 contained in (1) and (2) can be used, in the special case $d = 1$, to yield the ultimate algorithm.

iii) Humane algorithm. The essence of the algorithm most suitable to a human being comes from the following statement, which is an immediate consequence of Proposition 1.

PROPOSITION 3. *In the shortest path from* $\hat{i}^n$ *to* $\hat{j}^n$ $((i, j) \in \{0, 1, 2\}^2, i \neq j)$, *disc* 1 *is moved in the* μ*-th move if and only if* μ *is odd. It then moves in cyclic order*

from i *through* j *to* $i \circ j$, *if* n *is odd;*

from i *through* $i \circ j$ *to* j, *if* n *is even.*

Following Proposition 3 for odd moves, even moves are dictated by rule (0), so that the shortest path can be carried out rather speedy.

It has become obvious that the shortest path between perfect states can be made very transparent. It is even possible, by an inversion of Proposition 2, to construct a fast (i.e. $O(n)$) algorithm which decides if a given state $r \in T_n$ appears in the shortest path from $\hat{i}^n$ to $\hat{j}^n$ and, if it does, gives the number μ of moves it took to reach it starting from $\hat{i}^n$. This allows to continue the solution abandoned at a certain stage by somebody. Similarly, one can also determine μ if one finds a person who has died with a disc in his hand carrying through the shortest path. If, however, someone has committed an error during the effectuation, it is necessary to know how to solve $\mathfrak{P}1$.

1.3. PROBLEMS 1 AND 2

By Theorem 1, the existence of a shortest path from $s \in T_n$ to $t \in T_n$ is guaranteed.

Definition 5. Let $(s, t) \in T_n^2$. Then $\mu(s, t)$ denotes the length of the shortest path from s to t; if $t = \widehat{j}^n$, it will be written $\mu(s; j)$.

In this section for any pair (s, t) of regular states $\mu(s, t)$ will be determined and the shortest path(s) constructed. Finally, average values of μ will be deduced.

1.3.0. CONSTRUCTION OF THE SHORTEST PATHS BETWEEN REGULAR STATES

Although $\mathfrak{P}s$ 1 and 2 have been considered in literature (see Introduction), there is no proof of minimality in any of these papers, since everybody assumed that in a shortest path the largest disc moves only once (if at all). Example 1 shows the wrongness of this assumption. However, the following is true.

LEMMA 1. *Let* $p \in P_{n+1}(s, t)$ *be shortest. Then disc* $n + 1$ *moves*

o) *not at all if and only if* $s(n+1) = t(n+1)$,

i) *at most once if* s *or* t *is perfect,*

ii) *at most twice in general.*

Remark 3 and Definition 6. For $p \in P_n(s, t)$ define $-p \in T_n^{\mu_p+1}$ by

$$\forall \nu \in \{0, \ldots, \mu_p\}: -p_\nu = p_{\mu_p - \nu}.$$

It is easy to see that $-p \in P_n(t, s)$ and therefore it is clear that $-p$ is shortest iff p has this property. In view of this, part i of Lemma 1 will be proved for perfect t only.

Proof of Lemma 1. First observe that disc $n + 1$, once moved away from peg $k \in \{0, 1, 2\}$ during a shortest path p, will never come back to that peg, for suppose

$$\exists \mu', \mu'' \in \{1, \ldots, \mu_p\}, \ \mu' < \mu'': d_{\mu'}(p) = d_{\mu''}(p) = n + 1, \ i_{\mu'}(p) = j_{\mu''}(p) = k$$

and define a new path $\tilde{p}$ by deleting all the moves μ from p with $\mu' \leqslant \mu \leqslant \mu''$, $d_\mu(p) = n+1$, then $\tilde{p} \in P_{n+1}(s, t)$ (the position of disc $n+1$ does not limit the moves of the other discs!) and is shorter than p. This already proves o (the other part of o is trivial) and ii.

Now assume, for the proof of i, that disc $n+1$ moves twice in a shortest path p, in moves μ' and $\mu''(1 \leqslant \mu' < \mu'' \leqslant \mu_p)$ say. Then necessarily $\bar{p}_{\mu'} = \widehat{t(n+1)^n}$, $\bar{p}_{\mu''} = \widehat{s(n+1)^n}$ and, as t is supposed to be perfect, $\bar{p}_{\mu_p} = \widehat{t(n+1)^n}$, But this implies, by Theorem 2, $\mu_p - \mu'' \geqslant 2^n - 1$ and $\mu'' - 1 - \mu' \geqslant 2^n - 1$, such that $\mu_p \geqslant 2^{n+1} - 1 + \mu' \geqslant 2^{n+1}$, contradicting Theorem 1. □

With Lemma 1 on hand, it is now easy to construct shortest paths between regular states. Although the solution of $\mathfrak{P}2$ contains of course the solution of $\mathfrak{P}1$, it is convenient to state and prove the cases separately. The following definition will be useful.

Definition 7. For $r \in T_n$ and $j \in \{0, 1, 2\}$ let $r^j\colon \{0, \ldots, n\} \to \{0, 1, 2\}$ be defined by

$$(3) \qquad \begin{cases} r^j(n) = j\,, \\ \forall 0 \leqslant d < n\colon r^j(d) = r^j(d+1) \circ r(d+1)\,. \end{cases}$$

Note that $(3) \Leftrightarrow \forall 0 \leqslant d \leqslant n\colon r^j(d) = \left((-1)^{n-d}\{j + \sum_{c=d+1}^{n} (-1)^{n-c} r(c)\}\right) \bmod 3.$

THEOREM 3. *Let* $r \in T_n$ *and* $j \in \{0, 1, 2\}$. *Then*

$$\mu(r; j) = \sum_{\substack{d \in \{1, \ldots, n\} \\ r(d) \neq r^j(d)}} 2^{d-1}\,;$$

the shortest path from r *to* $\hat{j}$ *is unique and can be constructed in the following way:*

Beginning with r, *do:* (*for* $d = 1$ *to* n: (if $r(d) \neq r^j(d)$: (*move disc* d *from* $r(d)$ *to* $r^j(d)$ *and do* $p^{r^j(d-1), r^j(d); d-1}$))).

Definition 8. The shortest path from r to $\hat{j}^n$ will be denoted by $p^{r;j}$.

Proof of Theorem 3 by induction on n.

a) For $n = 0$ the statement is trivial.

b) If $r(n+1) = j$, then by Lemma 1o disc $n+1$ is not moved at all, and the shortest path from r to $\hat{j}^{n+1}$ is given by

$$\forall \nu \in \{0, ..., \mu(\bar{r}; j)\} : \overline{p_\nu^{r;j}} = p_\nu^{\bar{r};j}, p_\nu^{r;j}(n+1) = j;$$

the statements of the theorem follow easily using (3).

If $r(n+1) \neq j$, let $k := j \circ r(n+1)$; then by Lemma 1o and i, disc $n+1$ is moved exactly once and so the shortest path from r to $\hat{j}^{n+1}$ is given by

$$\forall \nu \in \{0, ..., \mu(\bar{r}; k)\} : \overline{p_\nu^{r;j}} = p_\nu^{\bar{r};k}, p_\nu^{r;j}(n+1) = r(n+1),$$

$$\forall \nu \in \{\mu(\bar{r}; k) + 1, ..., \mu(\bar{r}; k) + 2^n\} : \overline{p_\nu^{r;j}} = p_{\nu-\mu(\bar{r};k)-1}^{k,j;n}, p_\nu^{r;j}(n+1) = j,$$

from which again the statements of the theorem follow using (3). □

As an example, $\mu(r; 0) = 164$ for the r of Figure 1.

For presenting the solution of 𝔓2 it is, of course, no loss of generality to disregard the case of an empty **TH** and, in view of Lemma 1o, to assume that the largest disc is on different pegs in s and t. The following definition is needed.

Definition 9. Let $(s, t) \in T_{n+1}^2$, $s(n+1) \neq t(n+1)$. Then

$$\mu_1(s, t) := 1 + \mu(\bar{s}; s(n+1) \circ t(n+1)) + \mu(\bar{t}; s(n+1) \circ t(n+1)),$$
$$\mu_2(s, t) := 2^n + 1 + \mu(\bar{s}; t(n+1)) + \mu(\bar{t}; s(n+1)).$$

THEOREM 4. *Let* $(s, t) \in T_{n+1}^2$, $s(n+1) \neq t(n+1)$. *Then* $\mu(s, t) = \min\{\mu_1(s, t), \mu_2(s, t)\}$. *There are exactly two shortest paths from* s *to* t *if* $\mu_1(s, t) = \mu_2(s, t)$, *otherwise the shortest path is unique. The shortest path(s) can be constructed thus:*

if $\mu = \mu_1$: *Beginning with* s, *do* $p^{\bar{s}; s(n+1) \circ t(n+1)}$, *move disc* $n+1$ *from* $s(n+1)$ *to* $t(n+1)$, *do* $-p^{\bar{t}; s(n+1) \circ t(n+1)}$;

if $\mu = \mu_2$: *Beginning with* s, *do* $p^{\bar{s}; t(n+1)}$, *move disc* $n+1$ *from* $s(n+1)$ *to* $s(n+1) \circ t(n+1)$, *do* $p^{t(n+1), s(n+1); n}$, *move disc* $n+1$ *from* $s(n+1) \circ t(n+1)$ *to* $t(n+1)$, *do* $-p^{\bar{t}; s(n+1)}$.

Proof. It follows immediately from Lemma 1ii and Theorem 3 that the paths described in the statement of Theorem 4 are the only candidates for a shortest path from s to t. So one just has to choose the shorter of the two or both if their length is equal. □

Remark 4. It is easy to see that, using Theorems 4 and 3, it is possible to reduce 𝔓2 to the solution of 𝔓0, so that any of the algorithms in 1.2.1 can be employed to construct an algorithm for the solution of 𝔓2.

Although for any $(s, t) \in T_n^2$ the length $\mu(s, t)$ of the shortest path(s) from s to t can easily be calculated now, it is nevertheless interesting to know the average length of shortest paths explicitely. This will be examined in the following two subsections.

1.3.1. Discussion of the minimal length $\mu(r;j)$

A short glance at the graph of the **TH** (Figure 2) suggests the following results.

Proposition 4. *Let* $j \in \{0, 1, 2\}$. *Then* $\gamma_n := \sum_{r \in T_n} \mu(r;j) = 3^n \cdot \frac{2}{3}(2^n - 1)$.

Corollary 1. *The average length of shortest paths from regular to perfect states is* 2/3 *of the maximal length.*

The corollary follows immediately from Proposition 4, together with Theorems 0 and 2.

Proof of Proposition 4. $\gamma_0 = 0$ and Theorem 3 yields

$$\forall n \in \mathbf{N}_0 : \gamma_{n+1} = \sum_{\substack{r \in T_{n+1} \\ r(n+1) = j}} \mu(r;j) + \sum_{\substack{r \in T_{n+1} \\ r(n+1) \neq j}} \mu(r;j)$$

$$= \gamma_n + 2 \cdot 3^n \left(\frac{\gamma_n}{3^n} + 2^n\right) = 3\gamma_n + 2 \cdot 6^n .$$

Thus $\gamma_n = 2 \sum_{\kappa=0}^{n-1} 3^\kappa 6^{n-1-\kappa} = \frac{2}{3}(6^n - 3^n)$, where use has been made of

$$\text{(4)} \qquad \forall a \in \mathbf{R} \forall (a_n), (\alpha_n) \in \mathbf{R}^{\mathbf{N}_0} : ((\alpha_0 = 0 \wedge \forall n \in \mathbf{N}_0 : \alpha_{n+1} = a\alpha_n + a_n) \Leftrightarrow (\forall n \in \mathbf{N}_0 : \alpha_n = \sum_{\kappa=0}^{n-1} a^\kappa a_{n-1-\kappa}))$$

and

$$\text{(5)} \qquad \forall (a, b) \in \mathbf{R}^2, a \neq b \ \forall n \in \mathbf{N}_0 : \sum_{\kappa=0}^{n-1} b^\kappa a^{n-1-\kappa} = \frac{a^n - b^n}{a - b}. \qquad \square$$

The following is an interesting observation.

Proposition 5. *Let* $\mu \in \{0, ..., 2^n - 1\}$. *Then* $|\{r \in T_n; \mu(r;j) = \mu\}| = 2^{\beta(\mu)}$, *where* $\beta(\mu)$ *is the number of non-zero binary digits of* μ.

Remark 5. This is the population number of the μ-th level in the shortest path tree for $\hat{j}^n$, constructed for example for $j = 0$ (and $n = 3$) from Figure 2 by deleting all horizontal edges.

Proposition 5 is an easy consequence of the formula for $\mu(r;j)$ in Theorem 3 in view of (3). It can also serve as the base of an alternative proof of Proposition 4; this idea will be useful in the following subsection.

1.3.2. Discussion of the minimal length $\mu(s, t)$

The function $\mu(s, t)$ is much more puzzling than $\mu(r;j)$ because of the decision between μ_1 and μ_2 in Theorem 4. Although there seems to be no handy method, other than sheer computation, to find out, for given $(s, t) \in T_{n+1}^2$, which of the two is smaller, one can determine the number of events for each case.

PROPOSITION 6.

i) $| \{(s, t) \in T_{n+1}^2 ; s(n+1) \neq t(n+1), \mu_1(s, t) = \mu_2(s, t)\} | = \dfrac{6}{\sqrt{17}} (\Theta_+^n - \Theta_-^n)$,

ii) $| \{(s, t) \in T_{n+1}^2 ; s(n+1) \neq t(n+1), \mu_1(s, t) > \mu_2(s, t)\} |$

$$= \frac{3}{7} 9^n - \frac{3}{7} 2^n - \frac{3}{\sqrt{17}} (\Theta_+^n - \Theta_-^n),$$

iii) $| \{(s, t) \in T_{n+1}^2 ; s(n+1) \neq t(n+1), \mu_1(s, t) < \mu_2(s, t)\} |$

$$= \frac{39}{7} 9^n + \frac{3}{7} 2^n - \frac{3}{\sqrt{17}} (\Theta_+^n - \Theta_-^n);$$

here $\Theta_\pm := \dfrac{1}{2} (5 \pm \sqrt{17})$.

Remark 6. This is the first time, an irrational number enters, though implicitely, into considerations about the **TH**! By the way, $\sqrt{17}$ is one of the "oldest" irrationals, a proof for its incommensurability with unity being known in $-$ 398 to Theodorus of Cyrene (cp. [47, p. 141 ff]).

COROLLARY 2. *Asymptotically (for large n), the largest disc moves*

o) *not at all in* $\dfrac{1}{3}$,

i) *exactly once in* $\dfrac{13}{21}$,

ii) *exactly twice in* $\dfrac{1}{21}$

of all shortest paths between regular states.

This is an immediate consequence of Proposition 6 and the construction of the shortest path in Theorem 4.

The following functions will be useful in the proof of Proposition 6.

Definition 10. Let $\forall \mu \in \mathbf{Z}: z_n(\mu) = |\{r \in T_n;\, \mu(r;i) - \mu(r;j) = \mu\}|$; here (i,j) is any pair of distinct elements of $\{0, 1, 2\}$, and it is clear by symmetry that the definition does not depend on the specific pair employed.

The following lemma is a summary of properties of these functions.

LEMMA 2. o) $z_0(0) = 1, \forall \mu \in \mathbf{Z}\backslash\{0\}: z_0(\mu) = 0,$

$$\forall n \in \mathbf{N}_0\ \forall \mu \in \mathbf{Z}: z_{n+1}(\mu) = z_n(\mu-2^n) + z_n(\mu) + z_n(\mu+2^n);$$

i) $\forall \mu \in \mathbf{Z}: z_n(-\mu) = z_n(\mu), z_n(0) = 1, z_n(1) = n, z_n(2^n-1) = 1,$

$|\mu| \geqslant 2^n \Rightarrow z_n(\mu) = 0$;

ii) $\sum\limits_{\mu\in\mathbf{Z}} z_n(\mu) = 3^n, \sum\limits_{\mu\in\mathbf{N}} z_n(\mu) = \dfrac{1}{2}(3^n-1), \sum\limits_{\mu\in\mathbf{N}} \mu z_n(\mu) = \dfrac{1}{5}(6^n-1)$;

iii) *let* $x_n := \sum\limits_{\mu\in\mathbf{N}} z_n(\mu)\, z_n(2^n-\mu), y_n := \sum\limits_{\mu\in\mathbf{N}} z_n^2(\mu),$ *then*

$$x_n = \frac{1}{\sqrt{17}}(\Theta_+^n - \Theta_-^n),\ y_n = \frac{1}{4}\left(\left(1+\frac{1}{\sqrt{17}}\right)\Theta_+^n + \left(1-\frac{1}{\sqrt{17}}\right)\Theta_-^n - 2\right).$$

Proof. o) The statements about z_0 are trivial. The recursion relation is obtained from the fact

$$\mu(r;i) - \mu(r;j) = \begin{cases} \mu(\bar{r};i) - \mu(\bar{r};i\circ j) - 2^n, & \text{if} \quad r(n+1) = i, \\ \mu(\bar{r};i\circ j) - \mu(\bar{r};j) + 2^n, & \text{if} \quad r(n+1) = j, \\ \mu(\bar{r};j) - \mu(\bar{r};i), & \text{if} \quad r(n+1) = i\circ j, \end{cases}$$

which in turn follows from the construction in the proof of Theorem 3.

i) is proved by induction on n using o.

ii) is proved by induction on n using o and i.

iii) By o and i: $x_0 = 0, y_0 = 0, y_1 = 1$ and

$$\forall n \in \mathbf{N}_0: x_{n+1} = 2x_n + 2y_n + 1,\ y_{n+1} = 2x_n + 3y_n + 1,$$

such that $x_{n+1} = y_{n+1} - y_n$ and $y_{n+2} = 5y_{n+1} - 2y_n + 1$.

Defining $\eta_n := y_n + \frac{1}{2}$, the following recurrent sequence has to be calculated:

$$(6) \qquad \begin{cases} \eta_0 = \frac{1}{2}, \eta_1 = \frac{3}{2}, \\ \forall n \in \mathbf{N}_0 : \eta_{n+2} = 5\eta_{n+1} - 2\eta_n . \end{cases}$$

The ansatz $\tilde{\eta}_n = \Theta^n$ with a $\Theta \in \mathbf{R}$ leads to the solutions $\tilde{\eta}_n = \Theta_{\pm}^n$ of the recurrence relation, such that $\eta_n = \frac{1}{4}\left(\left(1 + \frac{1}{\sqrt{17}}\right)\Theta_+^n + \left(1 - \frac{1}{\sqrt{17}}\right)\Theta_-^n\right)$. The formulas for x_n and y_n are obtained from this by simple calculations. □

Proof of Proposition 6. i) Let $(s, t) \in T_{n+1}^2$, $s(n+1) \neq t(n+1)$, and define

$$\mu := \mu(\bar{s}; s(n+1) \circ t(n+1)) - \mu(\bar{s}; t(n+1)),$$
$$\tilde{\mu} := \mu(\bar{t}; s(n+1) \circ t(n+1)) - \mu(\bar{t}; s(n+1)).$$

Then $\mu_1(s, t) - \mu_2(s, t) = \mu + \tilde{\mu} - 2^n$ and

$$\mu_1(s, t) = \mu_2(s, t) \Leftrightarrow \mu, \tilde{\mu} \in \{1, \ldots, 2^n - 1\}, \tilde{\mu} = 2^n - \mu .$$

Thus, in view of the six different choices for $(s(n+1), t(n+1))$,

$$|\{(s, t) \in T_{n+1}^2 ; s(n+1) \neq t(n+1), \mu_1(s, t) = \mu_2(s, t)\}| = 6x_n ,$$

and Lemma 2 completes the proof of i.

ii) By a similar argument and with $\nu = 2^n - \tilde{\mu}$:

$$|\{(s, t) \in T_{n+1}^2 ; s(n+1) \neq t(n+1), \mu_1(s, t) > \mu_2(s, t)\}| = 6w_n ,$$

where $w_n := \sum_{\mu \in \mathbf{N}} \sum_{\nu=1}^{\mu-1} z_n(\mu) z_n(2^n - \nu)$. It is easy to see, using Lemma 2, that $w_0 = 0$ and $n \in \mathbf{N}_0 : w_{n+1} = 2w_n - y_n + \frac{1}{2}(3^{2n} - 1)$, which yields, by (4) and (5), the desired result.

iii) follows from

$$\begin{aligned} 3^{2(n+1)} &= |\{(s, t) \in T_{n+1}^2 ; s(n+1) = t(n+1)\}| \\ &+ |\{(s, t) \in T_{n+1}^2 ; s(n+1) \neq t(n+1), \mu_1(s, t) < \mu_2(s, t)\}| \\ &+ |\{(s, t) \in T_{n+1}^2 ; s(n+1) \neq t(n+1), \mu_1(s, t) > \mu_2(s, t)\}| \\ &+ |\{(s, t) \in T_{n+1}^2 ; s(n+1) \neq t(n+1), \mu_1(s, t) = \mu_2(s, t)\}| . \end{aligned}$$ □

By the same methods, the total and average number of moves in shortest paths between all regular states can be determined now.

PROPOSITION 7.

$$\delta_n := \sum_{(s,t)\in T_n^2} \mu(s,t) = \frac{466}{885} 18^n - \frac{1}{3} 9^n - \frac{3}{5} 3^n + \left(\frac{12}{59} + \frac{18}{1003}\sqrt{17}\right)\Theta_+^n + \left(\frac{12}{59} - \frac{18}{1003}\sqrt{17}\right)\Theta_-^n .$$

COROLLARY 3. *Asymptotically (for large n) the average length of shortest paths between regular states is* $\frac{466}{885}$ *of the maximal length.*

Again, this is an immediate consequence of Proposition 7 by Theorems 0 and 2.

Proof of Proposition 7. Clearly, $\delta_0 = 0$; let $n \in \mathbf{N}_0$; then

$$\begin{aligned} \delta_{n+1} &= \sum_{\substack{(s,t)\in T_{n+1}^2 \\ s(n+1)=t(n+1)}} \mu(s,t) + \sum_{\substack{(s,t)\in T_{n+1}^2 \\ s(n+1)\neq t(n+1)}} \mu(s,t) \\ &= 3\delta_n + \sum_{\substack{(s,t)\in T_{n+1}^2 \\ s(n+1)\neq t(n+1)}} \mu_1(s,t) - \sum_{\substack{(s,t)\in T_{n+1}^2 \\ s(n+1)\neq t(n+1) \\ \mu_1(s,t)>\mu_2(s,t)}} (\mu_1(s,t)-\mu_2(s,t)) . \end{aligned} \tag{7}$$

Let $(i,j) \in \{0,1,2\}^2$, $i \neq j$. Then

$$\begin{aligned} \sum_{\substack{(s,t)\in T_{n+1}^2 \\ s(n+1)\neq t(n+1)}} \mu_1(s,t) &= 6 \sum_{\substack{(s,t)\in T_{n+1}^2 \\ s(n+1)=i \\ t(n+1)=j}} \mu_1(s,t) = 6\cdot 3^{2n}\left(\frac{\gamma_n}{3^n} + 1 + \frac{\gamma_n}{3^n}\right) \\ &= 2\cdot 3^{2n}(2^{n+2}-1) . \end{aligned} \tag{8}$$

Using the same arguments as in the proof of Proposition 6, one gets

$$\sum_{\substack{(s,t)\in T_{n+1}^2 \\ s(n+1)\neq t(n+1) \\ \mu_1(s,t)>\mu_2(s,t)}} (\mu_1(s,t)-\mu_2(s,t)) = 6u_n, \quad \text{where}$$

$$u_n := \sum_{\mu\in\mathbf{N}} \sum_{\nu=1}^{\mu-1} (\mu-\nu) z_n(\mu) z_n(2^n-\nu) .$$

To calculate u_n, the following must be defined:

$$v_n := \sum_{\mu\in\mathbf{N}} \sum_{\nu=1}^{\mu-1} (\mu-\nu) z_n(\mu) z_n(\nu) .$$

Then the recursion relation holds:

$$(9)\quad \begin{cases} u_0 = v_0 = 0\,, \\ \forall n \in \mathbf{N}_0 : u_{n+1} = 2u_n + 2v_n + \frac{1}{5}(3^n+1)(6^n-1)\,, \\ v_{n+1} = 2u_n + 3v_n + \frac{1}{5}(6^n-1) + \frac{1}{2}6^n(3^n-1)\,; \end{cases}$$

this is proved with the aid of Lemma 2 and the facts

$$\forall n \in \mathbf{N}_0 : \sum_{\mu=1}^{2^n-1} \sum_{\nu=1}^{\mu-1} (\mu-\nu) z_n(2^n-\mu) z_n(2^n-\nu) = v_n\,,$$

$$\sum_{\mu=1}^{2^n-1} \sum_{\nu=1}^{\mu-1} (\mu-\nu) z_n(2^n-\mu) z_n(\nu) = u_n + 2^{n-2}(3^n-1)^2 - \frac{1}{5}(3^n-1)(6^n-1)\,,$$

which in turn follow from Lemma 2.

The solution of (9) is (analogously to (4))

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \sum_{\kappa=0}^{n-1} A^\kappa \begin{pmatrix} a_{n-1-\kappa} \\ b_{n-1-\kappa} \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix},$$

$$\forall \lambda \in \mathbf{N}_0 : a_\lambda = \frac{1}{5}(3^\lambda+1)(6^\lambda-1)\,,$$

$$b_\lambda = \frac{1}{5}(6^\lambda-1) + \frac{1}{2}6^\lambda(3^\lambda-1)\,.$$

Defining $\forall \kappa \in \mathbf{N}_0 : \eta_\kappa := \frac{1}{2}(A^\kappa)_{1,1} + \frac{1}{4}(A^\kappa)_{1,2}$, it turns out that $(A^{\kappa+1})_{1,2} = 2(\eta_{\kappa+1} - \eta_\kappa)$ and that $(\eta_\kappa)_{\kappa \in \mathbf{N}_0}$ fulfils (6). Thus $(A^\kappa)_{1,2} = \frac{2}{\sqrt{17}}(\Theta_+^\kappa - \Theta_-^\kappa)$ and $(A^\kappa)_{1,1} = \frac{1}{2}\left(\left(1 - \frac{1}{\sqrt{17}}\right)\Theta_+^\kappa + \left(1 + \frac{1}{\sqrt{17}}\right)\Theta_-^\kappa\right)$. A careful computation, with the aid of (5), yields

$$\forall n \in \mathbf{N}_0 : u_n = \frac{1}{59}18^n - \left(\frac{1}{118} + \frac{31}{2006}\sqrt{17}\right)\Theta_+^n - \left(\frac{1}{118} - \frac{31}{2006}\sqrt{17}\right)\Theta_-^n\,.$$

Inserting this and (8) into (7) leads to

$$\forall n \in \mathbf{N}_0 : \delta_{n+1} = 3\delta_n + \frac{466}{59}18^n - 2 \cdot 9^n + \left(\frac{3}{59} + \frac{93}{1003}\sqrt{17}\right)\Theta_+^n + \left(\frac{3}{59} - \frac{93}{1003}\sqrt{17}\right)\Theta_-^n\,,$$

and again with (4) and (5) the formula for δ_n is established. □