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THE TOWER OF HANOI

by Andreas M. HINZ

0. INTRODUCTION

About 100 years ago, the famous Tower of Hanoi made its first appearance in mathematical literature. The account of Allardice and Fraser [2] contains a literal repetition (in French) of an article by de Parville from the *Journal des Débats* for December 27th, 1883 (cp. [36]). In this earliest printed mention of the puzzle one can find the beautiful story of its legendary origin, involving brahmins moving 64 golden discs between diamond needles, and which has caused its popularity (for an early English version see Ball [3, p. 78 f]).

In a more prosaic diction, the Tower of Hanoi (**TH**) consists of three vertical pegs, fixed at the bottom, and a certain number n of circular discs of mutually different diameter, each disc being pierced in its center to allow it to be stacked on one of the pegs. Any distribution of the n discs among the three pegs is called a *state*. A state is called *regular*, if no disc lies on a smaller one and it is called *perfect*, if it is regular and all discs are stacked on the same peg. Figure 1 shows examples ($n = 8$).

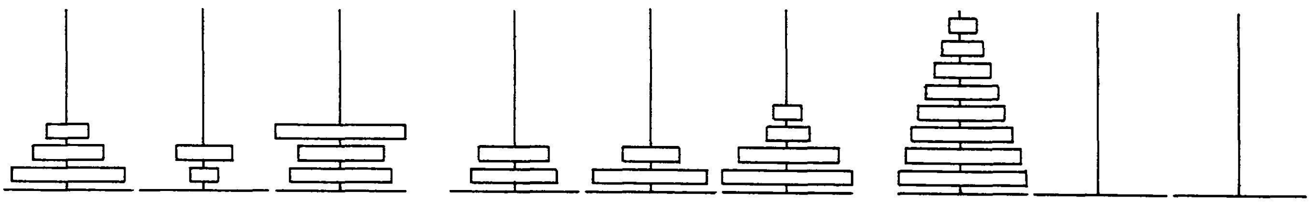

FIGURE 1.

A *(legal) move* is the transfer of exactly one disc from a peg to a different one, which apart from the mechanical restrictions (i.e. only the topmost disc on a peg can be moved, and it can only be stacked on an empty peg or onto the topmost disc of a peg) obeys the rule

(0) *No disc must ever be placed on a smaller one.*

Given two states σ *(initial state)* and τ *(final state)*, any finite sequence of moves which starts with σ and ends in τ will be called a *path* from σ to τ; the number of moves is the *length* of the path. The problem posed in the legend is to find, for $n = 64$, a shortest path from a perfect state to a different perfect state.

Actually, the puzzle had been turned out (with $n = 8$) in 1883 in Paris by a certain "N. Claus (de Siam)", anagram for "Lucas d'Amiens", as found out by de Parville. (At that time, France began her military involvement in Tonkin and Annam, so that names like "Hanoi" were in the headlines; this explains the name of the puzzle.) Edouard Lucas (1842-91) was a distinguished mathematician of his time (for his work see Harkin [24]), whose main achievements lie in number theory, but whose fame is based on this puzzle (see e.g. Gridgeman [21, p. 531 f]).

There is a parallel to Sir William Rowan Hamilton (1805-65) about whom it is told (see Graves [23, p. 55]) that the only money he ever earned with a piece of mathematics were 25 pounds he got for the copyright of the Icosian Game, the object of which was essentially to find what is nowadays called a Hamiltonian circuit on a dodecahedral graph. And in fact, it was pointed out by Crowe [10] that solving the problem of the **TH** yields a Hamiltonian circuit on an n-cube. This and the relation to the Chinese Ring puzzle is discussed in Afriat [1]. The link between the three puzzles is what Afriat calls, historically correct, the Gros code and what is now known as a Gray code of binary numbers.

The connection between the **TH** and binary numbers was of course familiar to Lucas. The cover plate (see [9, p. 128]) of the box in which his puzzle was sold, shows the name, written with bamboo leaves on a sheet of paper carried by a flying crane, of the legendary Chinese emperor Fo Hi (Fu Xi, -3rd millennium), to whom he attributes the invention of that number system (see [31, p. XVIII]; cp. [30, p. 149 ff]). With some more imagination one can even detect the last name of Pierre de Fermat (1601-65), written in the same manner starting on the disc of the rising sun. Lucas claims, on the printed leaflet [8] accompanying the puzzle, that it was found among the unedited writings of "l'illustre Mandarin FER-FER-TAM-TAM". In fact, Lucas had probably been sent to Rome in 1881, to prepare a couple of manuscripts for publication in Fermat's "Œuvres" (see Tannery [46, p. 9]), but the only connection of the **TH** to Fermat's papers may be the famous letter to Frenicle ([19, p. 205 f]), in which Fermat erroneously claims that $2^{64} + 1$, which he writes down in decimal representation, is prime. Lucas points out on the leaflet [8] that $2^{64} - 1$ is the length of the shortest path in the **TH** for $n = 64$ and that it would take more than five thousand million centuries to carry it out, moving one disc per second!

So it is very likely that Lucas himself is the inventor of the **TH** (cp. also [33, p. 55 ff]), and he might have been led to it in search for a generalization of the Chinese Ring puzzle to different number systems, for in [9] he mentions the possibility to represent these systems by modification of the **TH** and transformation of the rules. In fact, as can be seen using similar methods as in what will follow here, asking for a Hamiltonian path from one perfect state to another amounts in representing the number system of base 3 and, more generally, asking the same question in a puzzle with $3 + m$ pegs will lead to a representation of the system of base $3 + m\,(m \in \mathbf{N}_0)$.

The problem of finding a shortest path from a perfect state to another in a version of the **TH** with more than three pegs has been posed by Dudeney [15]. (Lucas published a five-peg version in a collection of puzzles in 1889, but the rules seem to be different (cp. [32]).) All the "solutions" which have appeared since Dudeney's challenge (see e.g. Stewart [45], Frame [20], Roth [41], Brousseau [6], Bendisch [4], Rohl and Gedeon [40]) are incomplete in that they construct short paths without proving them to be shortest (cp. Editorial Note following [45]). Only Cull and Ecklund [11], Wood [49], and Lunnon [34] point out that the problem is still open! This question partly motivated the present investigation.

Apart from that very interesting problem, the **TH** has experienced in recent years an astonishing revival. It appears in many textbooks on recursion (for which the classical **TH** is a very bad example!) in computer science (e.g. [39]), on algorithms (e.g. [37]), on discrete mathematics (e.g. [27]; [22] even starts with the **TH** on page 1!), on artificial intelligence (e.g. [7]), and on combinatorics (e.g. [5]). It is used in the discussion of complexity of algorithms (see Cull and Ecklund [12]) and even for psychological tests (see Simon [44], Matthes [35]). There are also many variants of it, but which change the rules and will therefore not be considered here.

Instead, the present investigation has been motivated by a second problem to be found on Lucas's leaflet [8], namely starting from any (possibly irregular) state to find a shortest path to a given perfect state. To solve this, five problems, depending on the type of regularity of the initial and final states, will be considered in detail: Find a shortest path for

Problem	initial state	final state
$\mathfrak{P}0$	perfect	perfect
$\mathfrak{P}1$	regular	perfect
$\mathfrak{P}2$	regular	regular
$\mathfrak{P}3$	irregular	perfect
$\mathfrak{P}4$	irregular	regular

$\mathfrak{P}0$, which is the original problem, was solved essentially in the first year after its coming out (see Allardice and Fraser [2], de Longchamps [29], Schoute [43]) by constructing a recursive solution and analysing it, but surprisingly, a complete proof of minimality has not been given until 1981 (Wood [49])! $\mathfrak{P}1$ has been considered in recent years in the computer science literature (see Walsh [48], Er [16, 18], Scarioni and Speranza [42], Pettorossi [38]), and an average minimal number of moves was given (Er [17]). Examples of $\mathfrak{P}2$ appear as problems in Domoryad [14, p. 75 f], and there is a remark on it in Er [17]. Wood [50] gives rules for a two-person game based on $\mathfrak{P}2$, but his theory of it is false. $\mathfrak{P}3$, which is Lucas's second problem, has been investigated by Lavallée [28], but there is no proof of minimality.

In Chapter 1, only regular states will be considered. Starting with an appropriate mathematical model (1.0), Section 1.1 will establish the existence of a solution to $\mathfrak{P}2$ and will give a sharp estimate for the minimal length. In 1.2, $\mathfrak{P}0$ will be completely unfolded by proving uniqueness of the minimal solution (1.2.0) and constructing this solution explicitely (1.2.1). Although these results can be found, more or less accurately, in many places, they will be given here to make clear the notions and to prepare Section 1.3, where $\mathfrak{P}s$ 1 and 2 will be solved. The main results are given in 1.3.0, where all minimal solutions are constructed and minimal lengths given. In 1.3.1 and 1.3.2 the average minimal numbers for $\mathfrak{P}1$ and $\mathfrak{P}2$, respectively, will be determined. Chapter 2 will be concerned with irregular states, i.e. $\mathfrak{P}s$ 3 and 4. After adjusting the mathematical model in 2.0, the existence of a solution for $\mathfrak{P}4$ can be established, and a sharp estimate for the minimal length will be given (2.1). Some tools for a recursive construction of solutions are provided in 2.2. For $\mathfrak{P}3$ one can also prove uniqueness of the minimal solution (2.3). Finally, Chapter 3 states some remarks on open problems, in particular the **TH** with more than three pegs.