

4. Relative de Rham homology

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **35 (1989)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

If we combine theorem (3.2) and the biduality theorem (2.1) we obtain what is usually known as the

(3.6) DE RHAM THEOREM. *Integration over smooth singular simplexes induces an isomorphism*

$$H^\bullet(X, \mathbf{C}) \xrightarrow{\sim} H_\infty^\bullet(X, \mathbf{C})$$

from de Rham cohomology to smooth singular cohomology.

4. RELATIVE DE RHAM HOMOLOGY

Let us start by some general remarks on the support of a compact p -chain T on a smooth n -dimensional manifold X . Since we can realize T as a section in the sheaf Ω^p^\vee the general sheaf theoretic notion of support applies: The *support* of T , $\text{Supp}(T)$ is the smallest closed subset Z of X , such that the restriction of T to $X - Z$ is zero.

(4.1) EXAMPLE. Integration over an oriented compact p -dimensional submanifold K with boundary defines a compact p -chain κ with $\text{Supp}(\kappa) = K$. From Stokes formula

$$\int_K d\omega = \int_{\partial K} \omega, \quad \omega \in \Gamma(X, \Omega^p),$$

we conclude that $\text{Supp}(b\kappa) = \partial K$.

Let us now consider the inclusion $j: U \rightarrow X$ of an open subset U of X . The induced map

$$j_*: D_p^c(U, \mathbf{C}) \rightarrow D_p^c(X, \mathbf{C}), \quad p \in \mathbf{N},$$

is injective since we may interpret j_* as "extension by zero" in the sheaf Ω_p^\vee , compare (2.5). A compact p -chain T on X belongs to the image of j_* if and only if $\text{Supp}(T) \subseteq U$. The complex $D_p^c(X, U; \mathbf{C})$ of *relative compact chains* is defined to fit into the following exact sequence

$$(4.2) \quad 0 \rightarrow D_p^c(U, \mathbf{C}) \xrightarrow{j_*} D_p^c(X, \mathbf{C}) \rightarrow D_p^c(X, U; \mathbf{C}) \rightarrow 0.$$

On this basis we can define the *relative de Rham* homology group

$$H_p(X, U; \mathbf{C}) = H_p D_p^c(X, U; \mathbf{C}), \quad p \in \mathbf{N}.$$

In more concrete terms we can describe this homology group as

$$(4.3) \quad \{Z \in D_p^c(X, \mathbf{C}) \mid \text{Supp}(bZ) \subseteq U\} \Big/ \begin{array}{l} \{bW \mid W \in D_{p+1}^c(X, \mathbf{C})\} \\ + \{Z \in D_p(X, \mathbf{C}) \mid \text{Supp}(Z) \subseteq U\} \end{array}$$

From the exact sequence (4.2) we deduce the homology sequence

$$(4.4) \quad \begin{array}{l} \rightarrow H_p^c(U, \mathbf{C}) \rightarrow H_p^c(X, \mathbf{C}) \rightarrow H_p^c(X, U; \mathbf{C}) \\ \rightarrow H_{p-1}^c(U, \mathbf{C}) \rightarrow H_{p-1}^c(X, \mathbf{C}) \rightarrow \end{array}$$

Let $f: X \rightarrow Y$ denote a smooth map, U an open subset of X and V an open subset of Y containing $f(U)$. Let us notice that

$$(4.5) \quad \text{Supp}(f_*T) \subseteq f(\text{Supp}(T)), \quad T \in D_p^c(X, \mathbf{C}).$$

These remarks make it evident, that de Rham homology is a covariant functor on the category of pairs consisting of a manifold and one of its open subspaces.

(4.6) *Excision.* Let Z be a closed subset of X and Y an open subset of X containing Z . The inclusion of $V = Y - Z$ in $U = X - Z$ induces an isomorphism

$$H^c(Y, V; \mathbf{C}) \xrightarrow{\sim} H^c(X, U; \mathbf{C}).$$

Proof. Let $i: Z \rightarrow X$ denote the inclusion. From the fact that $\Omega^{\bullet, \vee}$ consists of soft sheaves we deduce an exact sequence

$$0 \rightarrow \Gamma_c(U, \Omega^{\bullet, \vee}) \rightarrow \Gamma_c(X, \Omega^{\bullet, \vee}) \rightarrow \Gamma_c(Z, i^*\Omega^{\bullet, \vee}) \rightarrow 0$$

compare [5] III. 7.6. This allows us to make the identification

$$(4.7) \quad D^c(X, U; \mathbf{C}) \xrightarrow{\sim} \Gamma_c(Z, i^*\Omega^{\bullet, \vee}), \quad Z = X - U.$$

The expression on the right hand side is unchanged, when X is replaced by Y and U by V . Q.E.D.

(4.8) *Continuity.* Let (X_α) be an outward directed open covering of the manifold X . For any open subset U of X we have that

$$\lim_{\rightarrow \alpha} H^c(X_\alpha, U \cap X_\alpha; \mathbf{C}) = H^c(X, U; \mathbf{C})$$

Proof. As a consequence of the theorem of Borel-Heine, see possibly [5] III. 5.2, we find that

$$\lim_{\rightarrow} D^c(X_\alpha, \mathbf{C}) = D^c(X, \mathbf{C})$$

and similarly with X replaced by U and X_α replaced by $U \cap X_\alpha$. Using this and the exact sequence 4.2 we get that

$$\lim_{\rightarrow} D_c^c(X_\alpha, U \cap X_\alpha; \mathbf{C}) = D_c^c(X, U; \mathbf{C})$$

from which the result follows by passing to homology. Q.E.D.

Let us also notice that in case X is the disjoint union of a family (X_α) of open subsets we have that

$$(4.9) \quad \bigoplus_{\alpha} H_c^c(X_\alpha, U \cap X_\alpha; \mathbf{C}) \xrightarrow{\sim} H_c^c(X, U; \mathbf{C}).$$

5. STOKES FORMULA

Let us consider the open subset U of the n -dimensional smooth manifold X and the resulting exact sequences

$$(5.1) \quad \begin{aligned} &\rightarrow H_p^c(X, \mathbf{C}) \rightarrow H_p^c(X, U; \mathbf{C}) \xrightarrow{b} H_{p-1}^c(U, \mathbf{C}) \xrightarrow{j^*} H_{p-1}^c(X, \mathbf{C}) \rightarrow \\ &\leftarrow H^p(X, \mathbf{C}) \leftarrow H^p(X, U; \mathbf{C}) \xleftarrow{\partial} H^{p-1}(U, \mathbf{C}) \xleftarrow{j^*} H^{p-1}(X, \mathbf{C}) \leftarrow \end{aligned}$$

where the first is discussed in the previous section and the second is the sheaf cohomology sequence. The relative term in the second sequence is often written

$$(5.2) \quad H_Z^p(X, \mathbf{C}) = H^p(X, U; \mathbf{C}), \quad Z = X - U.$$

We can now extend the biduality theorem (2.1).

(5.3) THEOREM. *The cohomology sequence above is dual to the homology sequence. In particular we have a Stoke's formula*

$$\langle b\alpha, \omega \rangle = \langle \alpha, \partial\omega \rangle$$

for $\alpha \in H_p^c(X, U; \mathbf{C})$ and $\omega \in H^{p-1}(U, \mathbf{C})$.

Proof. The first sequence arises from the following short exact sequence of complexes, compare (4.2) and (4.7),

$$0 \rightarrow \Gamma_c(U, \Omega^{\bullet \vee}) \xrightarrow{j^*} \Gamma_c(X, \Omega^{\bullet \vee}) \rightarrow \Gamma_c(Z, \Omega^{\bullet \vee}) \rightarrow 0.$$

In order to calculate the second sequence we depart from the flabby resolution $\Omega^{\bullet \vee \vee}$ of \mathbf{R} established in the proof of the biduality theorem (2.1).