

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 35 (1989)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** CAUCHY RESIDUES AND DE RHAM HOMOLOGY  
**Autor:** Iversen, Birger  
**Kapitel:** 4. Relative de Rham homology  
**DOI:** <https://doi.org/10.5169/seals-57358>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 28.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

If we combine theorem (3.2) and the biduality theorem (2.1) we obtain what is usually known as the

(3.6) DE RHAM THEOREM. *Integration over smooth singular simplexes induces an isomorphism*

$$H^\bullet(X, \mathbf{C}) \xrightarrow{\sim} H_\infty^\bullet(X, \mathbf{C})$$

*from de Rham cohomology to smooth singular cohomology.*

#### 4. RELATIVE DE RHAM HOMOLOGY

Let us start by some general remarks on the support of a compact  $p$ -chain  $T$  on a smooth  $n$ -dimensional manifold  $X$ . Since we can realize  $T$  as a section in the sheaf  $\Omega^p^\vee$  the general sheaf theoretic notion of support applies: The *support* of  $T$ ,  $\text{Supp}(T)$  is the smallest closed subset  $Z$  of  $X$ , such that the restriction of  $T$  to  $X - Z$  is zero.

(4.1) EXAMPLE. Integration over an oriented compact  $p$ -dimensional submanifold  $K$  with boundary defines a compact  $p$ -chain  $\kappa$  with  $\text{Supp}(\kappa) = K$ . From Stokes formula

$$\int_K d\omega = \int_{\partial K} \omega, \quad \omega \in \Gamma(X, \Omega^p),$$

we conclude that  $\text{Supp}(b\kappa) = \partial K$ .

Let us now consider the inclusion  $j: U \rightarrow X$  of an open subset  $U$  of  $X$ . The induced map

$$j_*: D_p^c(U, \mathbf{C}) \rightarrow D_p^c(X, \mathbf{C}), \quad p \in \mathbf{N},$$

is injective since we may interpret  $j_*$  as "extension by zero" in the sheaf  $\Omega_p^\vee$ , compare (2.5). A compact  $p$ -chain  $T$  on  $X$  belongs to the image of  $j_*$  if and only if  $\text{Supp}(T) \subseteq U$ . The complex  $D_p^c(X, U; \mathbf{C})$  of *relative compact chains* is defined to fit into the following exact sequence

$$(4.2) \quad 0 \rightarrow D_p^c(U, \mathbf{C}) \xrightarrow{j_*} D_p^c(X, \mathbf{C}) \rightarrow D_p^c(X, U; \mathbf{C}) \rightarrow 0.$$

On this basis we can define the *relative de Rham* homology group

$$H_p(X, U; \mathbf{C}) = H_p D_p^c(X, U; \mathbf{C}), \quad p \in \mathbf{N}.$$

In more concrete terms we can describe this homology group as

$$(4.3) \quad \{Z \in D_p^c(X, \mathbf{C}) \mid \text{Supp}(bZ) \subseteq U\} / \left\{ \begin{array}{l} \{bW \mid W \in D_{p+1}^c(X, \mathbf{C})\} \\ + \{Z \in D_p(X, \mathbf{C}) \mid \text{Supp}(Z) \subseteq U\} \end{array} \right.$$

From the exact sequence (4.2) we deduce the homology sequence

$$(4.4) \quad \begin{array}{l} \rightarrow H_p^c(U, \mathbf{C}) \rightarrow H_p^c(X, \mathbf{C}) \rightarrow H_p^c(X, U; \mathbf{C}) \\ \rightarrow H_{p-1}^c(U, \mathbf{C}) \rightarrow H_{p-1}^c(X, \mathbf{C}) \rightarrow \end{array}$$

Let  $f: X \rightarrow Y$  denote a smooth map,  $U$  an open subset of  $X$  and  $V$  an open subset of  $Y$  containing  $f(U)$ . Let us notice that

$$(4.5) \quad \text{Supp}(f_*T) \subseteq f(\text{Supp}(T)), \quad T \in D_p^c(X, \mathbf{C}).$$

These remarks make it evident, that de Rham homology is a covariant functor on the category of pairs consisting of a manifold and one of its open subspaces.

(4.6) *Excision.* Let  $Z$  be a closed subset of  $X$  and  $Y$  an open subset of  $X$  containing  $Z$ . The inclusion of  $V = Y - Z$  in  $U = X - Z$  induces an isomorphism

$$H^c(Y, V; \mathbf{C}) \xrightarrow{\sim} H^c(X, U; \mathbf{C}).$$

*Proof.* Let  $i: Z \rightarrow X$  denote the inclusion. From the fact that  $\Omega^{\bullet, \vee}$  consists of soft sheaves we deduce an exact sequence

$$0 \rightarrow \Gamma_c(U, \Omega^{\bullet, \vee}) \rightarrow \Gamma_c(X, \Omega^{\bullet, \vee}) \rightarrow \Gamma_c(Z, i^*\Omega^{\bullet, \vee}) \rightarrow 0$$

compare [5] III. 7.6. This allows us to make the identification

$$(4.7) \quad D^c(X, U; \mathbf{C}) \xrightarrow{\sim} \Gamma_c(Z, i^*\Omega^{\bullet, \vee}), \quad Z = X - U.$$

The expression on the right hand side is unchanged, when  $X$  is replaced by  $Y$  and  $U$  by  $V$ . Q.E.D.

(4.8) *Continuity.* Let  $(X_\alpha)$  be an outward directed open covering of the manifold  $X$ . For any open subset  $U$  of  $X$  we have that

$$\lim_{\rightarrow \alpha} H^c(X_\alpha, U \cap X_\alpha; \mathbf{C}) = H^c(X, U; \mathbf{C})$$

*Proof.* As a consequence of the theorem of Borel-Heine, see possibly [5] III. 5.2, we find that

$$\lim_{\rightarrow} D^c(X_\alpha, \mathbf{C}) = D^c(X, \mathbf{C})$$

and similarly with  $X$  replaced by  $U$  and  $X_\alpha$  replaced by  $U \cap X_\alpha$ . Using this and the exact sequence 4.2 we get that

$$\lim_{\rightarrow} D_c^c(X_\alpha, U \cap X_\alpha; \mathbf{C}) = D_c^c(X, U; \mathbf{C})$$

from which the result follows by passing to homology. Q.E.D.

Let us also notice that in case  $X$  is the disjoint union of a family  $(X_\alpha)$  of open subsets we have that

$$(4.9) \quad \bigoplus_{\alpha} H_c^c(X_\alpha, U \cap X_\alpha; \mathbf{C}) \xrightarrow{\sim} H_c^c(X, U; \mathbf{C}).$$

## 5. STOKES FORMULA

Let us consider the open subset  $U$  of the  $n$ -dimensional smooth manifold  $X$  and the resulting exact sequences

$$(5.1) \quad \begin{array}{ccccccc} \rightarrow & H_p^c(X, \mathbf{C}) & \rightarrow & H_p^c(X, U; \mathbf{C}) & \xrightarrow{b} & H_{p-1}^c(U, \mathbf{C}) & \xrightarrow{j_*} & H_{p-1}^c(X, \mathbf{C}) & \rightarrow \\ \leftarrow & H^p(X, \mathbf{C}) & \leftarrow & H^p(X, U; \mathbf{C}) & \xleftarrow{\partial} & H^{p-1}(U, \mathbf{C}) & \xleftarrow{j^*} & H^{p-1}(X, \mathbf{C}) & \leftarrow \end{array}$$

where the first is discussed in the previous section and the second is the sheaf cohomology sequence. The relative term in the second sequence is often written

$$(5.2) \quad H_Z^p(X, \mathbf{C}) = H^p(X, U; \mathbf{C}), \quad Z = X - U.$$

We can now extend the biduality theorem (2.1).

(5.3) THEOREM. *The cohomology sequence above is dual to the homology sequence. In particular we have a Stoke's formula*

$$\langle b\alpha, \omega \rangle = \langle \alpha, \partial\omega \rangle$$

for  $\alpha \in H_p^c(X, U; \mathbf{C})$  and  $\omega \in H^{p-1}(U, \mathbf{C})$ .

*Proof.* The first sequence arises from the following short exact sequence of complexes, compare (4.2) and (4.7),

$$0 \rightarrow \Gamma_c(U, \Omega^{\bullet \vee}) \xrightarrow{j_*} \Gamma_c(X, \Omega^{\bullet \vee}) \rightarrow \Gamma_c(Z, \Omega^{\bullet \vee}) \rightarrow 0.$$

In order to calculate the second sequence we depart from the flabby resolution  $\Omega^{\bullet \vee \vee}$  of  $\mathbf{R}$  established in the proof of the biduality theorem (2.1).