THE TOWER OF HANOI

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0. INTRODUCTION

About 100 years ago, the famous Tower of Hanoi made its first appearance in mathematical literature. The account of Allardice and Fraser [2] contains a literal repetition (in French) of an article by de Parville from the *Journal des Débats* for December 27th, 1883 (cp. [36]). In this earliest printed mention of the puzzle one can find the beautiful story of its legendary origin, involving brahmins moving 64 golden discs between diamond needles, and which has caused its popularity (for an early English version see Ball [3, p. 78 f]).

In a more prosaic diction, the Tower of Hanoi (**TH**) consists of three vertical pegs, fixed at the bottom, and a certain number n of circular discs of mutually different diameter, each disc being pierced in its center to allow it to be stacked on one of the pegs. Any distribution of the n discs among the three pegs is called a *state*. A state is called *regular*, if no disc lies on a smaller one and it is called *perfect*, if it is regular and all discs are stacked on the same peg. Figure 1 shows examples ($n = 8$).

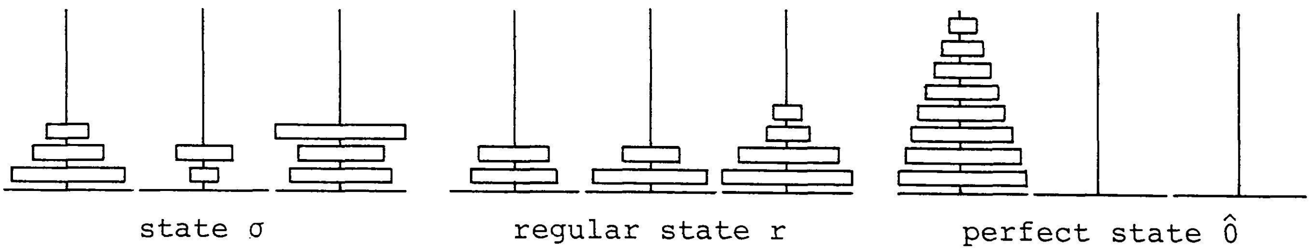

FIGURE 1.

A *(legal) move* is the transfer of exactly one disc from a peg to a different one, which apart from the mechanical restrictions (i.e. only the topmost disc on a peg can be moved, and it can only be stacked on an empty peg or onto the topmost disc of a peg) obeys the rule

(0) *No disc must ever be placed on a smaller one.*

Given two states σ *(initial state)* and τ *(final state)*, any finite sequence of moves which starts with σ and ends in τ will be called a *path* from σ to τ; the number of moves is the *length* of the path. The problem posed in the legend is to find, for $n = 64$, a shortest path from a perfect state to a different perfect state.

Actually, the puzzle had been turned out (with $n = 8$) in 1883 in Paris by a certain "N. Claus (de Siam)", anagram for "Lucas d'Amiens", as found out by de Parville. (At that time, France began her military involvement in Tonkin and Annam, so that names like "Hanoi" were in the headlines; this explains the name of the puzzle.) Edouard Lucas (1842-91) was a distinguished mathematician of his time (for his work see Harkin [24]), whose main achievements lie in number theory, but whose fame is based on this puzzle (see e.g. Gridgeman [21, p. 531 f]).

There is a parallel to Sir William Rowan Hamilton (1805-65) about whom it is told (see Graves [23, p. 55]) that the only money he ever earned with a piece of mathematics were 25 pounds he got for the copyright of the Icosian Game, the object of which was essentially to find what is nowadays called a Hamiltonian circuit on a dodecahedral graph. And in fact, it was pointed out by Crowe [10] that solving the problem of the **TH** yields a Hamiltonian circuit on an n-cube. This and the relation to the Chinese Ring puzzle is discussed in Afriat [1]. The link between the three puzzles is what Afriat calls, historically correct, the Gros code and what is now known as a Gray code of binary numbers.

The connection between the **TH** and binary numbers was of course familiar to Lucas. The cover plate (see [9, p. 128]) of the box in which his puzzle was sold, shows the name, written with bamboo leaves on a sheet of paper carried by a flying crane, of the legendary Chinese emperor Fo Hi (Fu Xi, -3rd millennium), to whom he attributes the invention of that number system (see [31, p. XVIII]; cp. [30, p. 149 ff]). With some more imagination one can even detect the last name of Pierre de Fermat (1601-65), written in the same manner starting on the disc of the rising sun. Lucas claims, on the printed leaflet [8] accompanying the puzzle, that it was found among the unedited writings of "l'illustre Mandarin FER-FER-TAM-TAM". In fact, Lucas had probably been sent to Rome in 1881, to prepare a couple of manuscripts for publication in Fermat's "Œuvres" (see Tannery [46, p. 9]), but the only connection of the **TH** to Fermat's papers may be the famous letter to Frenicle ([19, p. 205 f]), in which Fermat erroneously claims that $2^{64} + 1$, which he writes down in decimal representation, is prime. Lucas points out on the leaflet [8] that $2^{64} - 1$ is the length of the shortest path in the **TH** for $n = 64$ and that it would take more than five thousand million centuries to carry it out, moving one disc per second!

So it is very likely that Lucas himself is the inventor of the **TH** (cp. also [33, p. 55 ff]), and he might have been led to it in search for a generalization of the Chinese Ring puzzle to different number systems, for in [9] he mentions the possibility to represent these systems by modification of the **TH** and transformation of the rules. In fact, as can be seen using similar methods as in what will follow here, asking for a Hamiltonian path from one perfect state to another amounts in representing the number system of base 3 and, more generally, asking the same question in a puzzle with $3 + m$ pegs will lead to a representation of the system of base $3 + m\,(m \in \mathbf{N}_0)$.

The problem of finding a shortest path from a perfect state to another in a version of the **TH** with more than three pegs has been posed by Dudeney [15]. (Lucas published a five-peg version in a collection of puzzles in 1889, but the rules seem to be different (cp. [32]).) All the "solutions" which have appeared since Dudeney's challenge (see e.g. Stewart [45], Frame [20], Roth [41], Brousseau [6], Bendisch [4], Rohl and Gedeon [40]) are incomplete in that they construct short paths without proving them to be shortest (cp. Editorial Note following [45]). Only Cull and Ecklund [11], Wood [49], and Lunnon [34] point out that the problem is still open! This question partly motivated the present investigation.

Apart from that very interesting problem, the **TH** has experienced in recent years an astonishing revival. It appears in many textbooks on recursion (for which the classical **TH** is a very bad example!) in computer science (e.g. [39]), on algorithms (e.g. [37]), on discrete mathematics (e.g. [27]; [22] even starts with the **TH** on page 1!), on artificial intelligence (e.g. [7]), and on combinatorics (e.g. [5]). It is used in the discussion of complexity of algorithms (see Cull and Ecklund [12]) and even for psychological tests (see Simon [44], Matthes [35]). There are also many variants of it, but which change the rules and will therefore not be considered here.

Instead, the present investigation has been motivated by a second problem to be found on Lucas's leaflet [8], namely starting from any (possibly irregular) state to find a shortest path to a given perfect state. To solve this, five problems, depending on the type of regularity of the initial and final states, will be considered in detail: Find a shortest path for

Problem	initial state	final state
𝔓0	perfect	perfect
𝔓1	regular	perfect
𝔓2	regular	regular
𝔓3	irregular	perfect
𝔓4	irregular	regular

𝔓0, which is the original problem, was solved essentially in the first year after its coming out (see Allardice and Fraser [2], de Longchamps [29], Schoute [43]) by constructing a recursive solution and analysing it, but surprisingly, a complete proof of minimality has not been given until 1981 (Wood [49])! 𝔓1 has been considered in recent years in the computer science literature (see Walsh [48], Er [16, 18], Scarioni and Speranza [42], Pettorossi [38]), and an average minimal number of moves was given (Er [17]). Examples of 𝔓2 appear as problems in Domoryad [14, p. 75 f], and there is a remark on it in Er [17]. Wood [50] gives rules for a two-person game based on 𝔓2, but his theory of it is false. 𝔓3, which is Lucas's second problem, has been investigated by Lavallée [28], but there is no proof of minimality.

In Chapter 1, only regular states will be considered. Starting with an appropriate mathematical model (1.0), Section 1.1 will establish the existence of a solution to 𝔓2 and will give a sharp estimate for the minimal length. In 1.2, 𝔓0 will be completely unfolded by proving uniqueness of the minimal solution (1.2.0) and constructing this solution explicitely (1.2.1). Although these results can be found, more or less accurately, in many places, they will be given here to make clear the notions and to prepare Section 1.3, where 𝔓*s* 1 and 2 will be solved. The main results are given in 1.3.0, where all minimal solutions are constructed and minimal lengths given. In 1.3.1 and 1.3.2 the average minimal numbers for 𝔓1 and 𝔓2, respectively, will be determined. Chapter 2 will be concerned with irregular states, i.e. 𝔓*s* 3 and 4. After adjusting the mathematical model in 2.0, the existence of a solution for 𝔓4 can be established, and a sharp estimate for the minimal length will be given (2.1). Some tools for a recursive construction of solutions are provided in 2.2. For 𝔓3 one can also prove uniqueness of the minimal solution (2.3). Finally, Chapter 3 states some remarks on open problems, in particular the **TH** with more than three pegs.

1. Regular states

This chapter will develop an essentially complete theory of finding minimal paths between regular states of the **TH**. It starts with an appropriate formal setting.

1.0. Mathematical model

The *pegs* will be denoted by an $i \in \{0, 1, 2\}$, the *discs* by $d \in \{1, \ldots, n\}$ in natural order of increasing diameter; $n \in \mathbf{N}_0$ throughout, if not otherwise stated.

Definition 0. $T_n := \{r : \{1, \ldots, n\} \to \{0, 1, 2\}\}$. An $r \in T_n$ will also be written as $[r(1), \ldots, r(n)]$.

It is evident that any regular state of the **TH** is completely described by one and only one $r \in T_n$ and that any $r \in T_n$ can be interpreted as one and only one regular state of the **TH**. So it follows immediately by induction:

THEOREM 0. *The number of regular states of the* **TH** *with* $n \in \mathbf{N}_0$ *discs is* 3^n.

Definition 1. i) A pair $(r_0, r_1) \in T_n^2$ is a *(legal) move* (of disc d from peg i to peg j), iff

$$\exists (i,j) \in \{0,1,2\}^2, i \neq j : \left(r_0^{-1}(\{i\}) \neq \emptyset \wedge \left(r_0^{-1}(\{j\}) = \emptyset \vee d := \min r_0^{-1}(\{i\}) < \min r_0^{-1}(\{j\})\right) \wedge \left(r_1(d) = j \wedge \forall c \in \{1, \ldots, n\} \setminus \{d\} : r_1(c) = r_0(c)\right)\right).$$

ii) For any pair $(s, t) \in T_n^2$ let

$$P_n(s,t) := \{p \in \bigcup_{\nu=0}^{\infty} T_n^{\nu+1} ; p_0 = s, p_{\mu_p} = t \wedge \forall \mu \in \{1, \ldots, \mu_p\} : (p_{\mu-1}, p_\mu) \text{ is a move}\}$$

where $\mu_p := \operatorname{ind}(p)$.

A $p \in P_n(s, t)$ is called a *path* from s to t; μ_p is the *length* of p.

With this adequate formal model, it is now possible to treat $\mathfrak{P}$s 0 to 2, namely to find shortest paths between regular states. The following notions will frequently be used:

Definition 2. i) For any $r \in T_{n+1} : \bar{r} := r \mid \{1, \ldots, n\}\ (\in T_n)$.

ii) For $(i, j) \in \{0, 1, 2\}^2$:

$$i \circ j := \begin{cases} i\,, & \text{if} \quad i = j\,; \\ k \in \{0, 1, 2\} \backslash \{i, j\}\,, & \text{if} \quad i \neq j\,. \end{cases}$$

(Note that $i \circ j = \big(-(i+j)\big) \bmod 3$.)

iii) For $i \in \{0, 1, 2\} : \widehat{i^n} := [i, ..., i] \in T_n$. (These are the perfect states.)

As pointed out by Er [17], it is often convenient to regard the **TH** as a graph, the vertices of which being the regular states and in which the edges are formed by the legal moves. It will turn out that this graph is planar, simple, and connected. An example ($n=3$) is given in Figure 2:

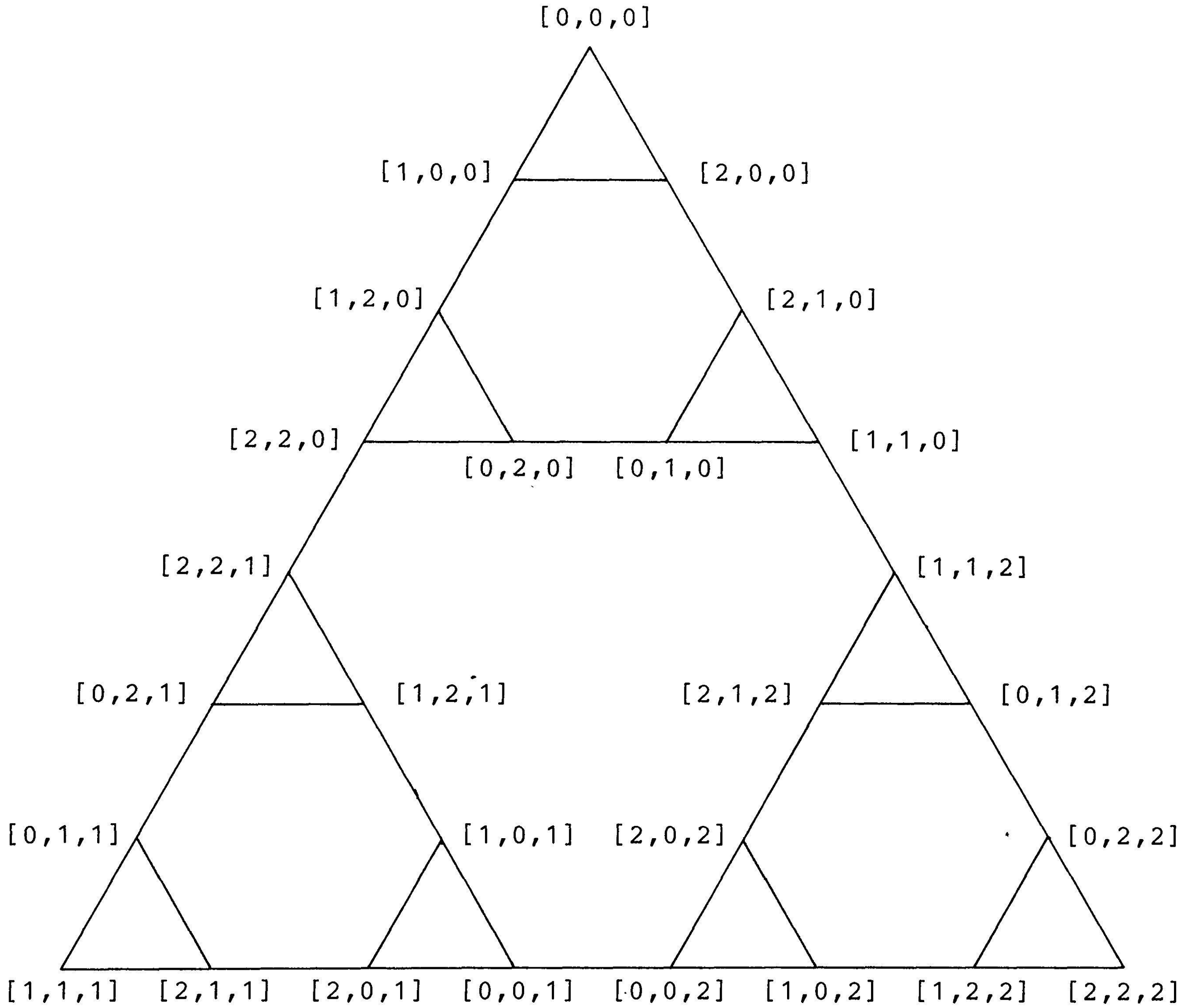

FIGURE 2.

1.1. Existence of a shortest path between two regular states and an upper bound for its length

To establish the sheer existence of a shortest path from s to t it suffices to show that $P_n(s, t) \neq \emptyset$.

THEOREM 1. *For any pair* (s, t) *of regular states there is a (shortest) path from* s *to* t *with length less than or equal to* $2^n - 1$, *where* n *is the number of discs involved.*

Proof by induction. a) The case $n = 0$ is trivial.

b) Let $(s, t) \in T^2_{n+1}$.

If $s(n+1) = t(n+1)$, let $\tilde{p} \in P_n(\bar{s}, \bar{t})$ with $\mu_{\tilde{p}} \leqslant 2^n - 1$, and define $p \in T^{\mu_p+1}_{n+1}$ by $\mu_p = \mu_{\tilde{p}}(\leqslant 2^{n+1}-1)$ and $\forall \nu \in \{0, ..., \mu_p\}: \bar{p}_\nu = \tilde{p}_\nu$, $p_\nu(n+1) = s(n+1)$. It is easy to see that $p \in P_{n+1}(s, t)$.

If $s(n+1) \neq t(n+1)$, let $i := s(n+1) \circ t(n+1)$, $\tilde{p} \in P_n(\bar{s}, \hat{i})$ and $\tilde{q} \in P_n(\hat{i}, \bar{t})$ with $\mu_{\tilde{p}}, \mu_{\tilde{q}} \leqslant 2^n - 1$. Define $p \in T^{\mu_p+1}_{n+1}$ by $\mu_p = \mu_{\tilde{p}} + \mu_{\tilde{q}} + 1(\leqslant 2^{n+1}-1)$ and

$$\forall \nu \in \{0, ..., \mu_{\tilde{p}}\}: \bar{p}_\nu = \tilde{p}_\nu, p_\nu(n+1) = s(n+1),$$

$$\forall \nu \in \{\mu_{\tilde{p}} + 1, ..., \mu_p\}: \bar{p}_\nu = \tilde{q}_{\nu-\mu_{\tilde{p}}-1}, p_\nu(n+1) = t(n+1).$$

Then $p \in P_{n+1}(s, t)$. □

Remark 1. The proof of Theorem 1 is constructive in that it allows to determine a path from s to t recursively.

In all papers mentioned in the introduction and dealing with 𝔓*s* 0 to 2, except those by Er [17] and Wood [49], it has been assumed that the shortest path is uniquely defined by this construction. But neither is the shortest path unique in general, nor does the construction always produce a shortest path, even if one chooses $\tilde{p}$ and $\tilde{q}$ minimal!

Example 1. a) Let $n = 2$, $s = [0, 1]$, $t = [1, 0]$,

b) Let $n = 3$, $s = [0, 0, 1]$, $t = [1, 1, 0]$.

Then a look at the graph in Figure 2 immediately shows that ([0, 1], [2, 1], [2, 0], [1, 0]) and ([0, 1], [0, 2], [1, 2], [1, 0]) are both shortest paths for a), and for b) the construction of Theorem 1 leads to the path ([0, 0, 1], [1, 0, 1], [1, 2, 1], [2, 2, 1], [2, 2, 0], [0, 2, 0], [0, 1, 0], [1, 1, 0]) of length 7, while ([0, 0, 1], [0, 0, 2], [2, 0, 2], [2, 1, 2], [1, 1, 2], [1, 1, 0]) of length 5 is shortest.

Er [17] refers to symmetry properties of the graph to establish uniqueness for 𝔓*s* 0 and 1. In [49], Wood felt the obligation to prove that the path of Theorem 1 is shortest for $s = \hat{i}$, $t = \hat{j}$ (see Section 1.2 below), but in [50], he made the mistake to assume its minimality in the case of general s and t, an error repeated by Cull and Gerety [13] (obviously, **TH** is really hard!). This problem will be treated correctly in Section 1.3.

1.2. Perfect states

This section will leave no secret about the classical 𝔓0. The essential step is to establish uniqueness of the shortest path between perfect states.

1.2.0. Uniqueness and length of the classical solution

Theorem 2. *For any two distinct pegs* i *and* j, *there is exactly one shortest path from* $\hat{i}^n$ *to* $\hat{j}^n$; *its length is* $2^n - 1$.

Proof. It will be shown by induction that

$$\forall (i,j) \in \{0,1,2\}^2,\ i \neq j\ \exists_1\ p \in P_n(\hat{i}, \hat{j}): \mu_p = 2^n - 1 \text{ is minimal}.$$

a) The case $n = 0$ is trivial.

b) Let $p \in P_{n+1}(\hat{i}, \hat{j})$ be shortest. As $i \neq j$, disc $n+1$ must be moved at least once. Before the first move of disc $n+1$, from i to $k \neq i$ say, discs 1 to n have to be brought from i to $i \circ k$ by the rules of a legal move of $n+1$; this is equivalent to a path from $\hat{i}^n$ to $\widehat{i \circ k}^n$, which takes at least $2^n - 1$ moves of discs 1 to n.

After the last move of $n+1$, from $l \neq j$ to j say, discs 1 to n must be brought from $l \circ j$ to j, which again takes at least $2^n - 1$ moves. So $\mu_p \geqslant 2^{n+1} - 1$.

As $\mu_p \leqslant 2^{n+1} - 1$ by Theorem 1, it follows that disc $n+1$ moves exactly once, i.e. $k = l = i \circ j$, which implies uniqueness of p too. □

Definition 3. The shortest path from $\hat{i}^n$ to $\hat{j}^n$ will be denoted by $p^{i,j;n}$.

Remark 2. Theorem 2 shows that the bound on the length of a shortest path in Theorem 1 is sharp.

1.2.1. Construction of the shortest path between two perfect states

A large part of the interest the **TH** has raised in recent years, stems from the discussion, mostly among computer scientists, which algorithm for the realization of the shortest path between perfect states is the "best". The right question is, of course: "best for what?". Four constructions will be given here, each of which suitable for a different situation. The recursive solution in o, already to be found in [8], is the backbone of the theory and fits best into textbooks on recursion. The iterative solution in i (cp. [28]), or some derived version of it, can best be used to make a computer do the **TH**. It also immediately leads to a description of the shortest path in just one formula; this algorithm ii (cp. Hering [25]) can make a parallel computer write down the solution more or less "at once". As man's mental quickness is much more limited, these algorithms are not suited to him. But there is another iterative variant iii, developped essentially in [43], allowing a human being to carry out the shortest path at a rate of about one move per second, a speed consistent with the traditional assumption of many authors.

o) Recursive algorithm. An immediate consequence of the proof of Theorem 2 is

PROPOSITION 0. *Let* $(i, j) \in \{0, 1, 2\}^2, i \neq j$. *Then*

a) $p^{i,j;0} = (\emptyset)$.

b) *For any* n, $p^{i,j;n+1}$ *is given by*

$$\forall \nu \in \{0, \ldots, 2^n - 1\} : \overline{p_\nu^{i,j;n+1}} = p_\nu^{i,i\circ j;n}, p_\nu(n+1) = i;$$

$$\forall \nu \in \{2^n, \ldots, 2^{n+1} - 1\} : \overline{p_\nu^{i,j;n+1}} = p_{\nu-2^n}^{i\circ j,j;n}, p_\nu(n+1) = j.$$

It is clear that this algorithm is of little practical interest (for large n, a huge amount of memory is needed just to do the first move!), but it serves as theoretical base for the following algorithms.

i) Iterative algorithm. This algorithm tells for the μ-th move of the shortest path which disc to move and determines its initial and final peg during that move.

Definition 4. Let $p \in P_n(s, t)$, $\mu \in \{1, \ldots, \mu_p\}$. Then

o) $(p_{\mu-1}, p_\mu)$ is called the μ-th move of p;

i) $d_\mu(p) :=$ disc moved in the μ-th move of p;

ii) $i_\mu(p) :=$ peg from which $d_\mu(p)$ is moved in the μ-th move of p;

iii) $j_\mu(p) :=$ peg to which $d_\mu(p)$ is moved in the μ-th move of p.

These notions are well-defined in view of Definition 1.

PROPOSITION 1. *Let* $(i, j) \in \{0, 1, 2\}^2, i \neq j$. *Then for any* $\mu \in \{1, \ldots, 2^n - 1\}$:

o) $d := d_\mu(p^{i,j;n}) = \min\{c \in \{1, \ldots, n\}; 2^c \nmid \mu\}$;

i) $i_\mu(p^{i,j;n}) = \left(\left(\frac{\mu}{2^d} - \frac{1}{2}\right)(j-i)((n-d) \bmod 2 + 1) + i\right) \bmod 3$;

ii) $j_\mu(p^{i,j;n}) = \left(\left(\frac{\mu}{2^d} + \frac{1}{2}\right)(j-i)\left((n-d) \bmod 2+1\right)+i\right) \bmod 3\,.$

Proof by induction on n.

a) For $n = 0$, the statement is trivial.

b) Proposition 0b yields: For $\mu \in \{1, ..., 2^n - 1\}$:

$$\begin{aligned} d := d_\mu(p^{i,j;n+1}) = d_\mu(p^{i,i\circ j;n}) &= \min\{c \in \{1, ..., n\}; 2^c \nmid \mu\} \\ &= \min\{c \in \{1, ..., n+1\}; 2^c \nmid \mu\}\,, \end{aligned}$$

$$\begin{aligned} i_\mu(p^{i,j;n+1}) = i_\mu(p^{i,i\circ j;n}) &= \left(\left(\frac{\mu}{2^d} - \frac{1}{2}\right)((i\circ j)-i)\left((n-d) \bmod 2+1\right)+i\right) \bmod 3 \\ &= \left(\left(\frac{\mu}{2^d} - \frac{1}{2}\right)(i-j)\left((n-d) \bmod 2+1\right)+i\right) \bmod 3 \\ &= \left(\left(\frac{\mu}{2^d} - \frac{1}{2}\right)(j-i)\left(((n+1)-d) \bmod 2+1\right)+i\right) \bmod 3\,, \end{aligned}$$

$j_\mu(p^{i,j;n+1}) = \ldots.$ (analogously);

for $\mu = 2^n$: $d = n+1$, $i_\mu(p^{i,j;n+1}) = i$, $j_\mu(p^{i,j;n+1}) = j$;
for $\mu \in \{2^n + 1, ..., 2^{n+1} - 1\}$:

$$\begin{aligned} d = d_{\mu-2^n}(p^{i\circ j,j;n}) &= \min\{c \in \{1, ..., n\}; 2^c \nmid \mu - 2^n\} \\ &= \min\{c \in \{1, ..., n+1\}; 2^c \nmid \mu\}\,, \end{aligned}$$

$$\begin{aligned} i_\mu(p^{i,j;n+1}) &= i_{\mu-2^n}(p^{i\circ j,j;n}) \\ &= \left(\left(\frac{\mu-2^n}{2^d} - \frac{1}{2}\right)(j-(i\circ j))\left((n-d) \bmod 2+1\right)+(i\circ j)\right) \bmod 3 \\ &= \left(\left(\frac{\mu}{2^d} - \frac{1}{2}\right)((i\circ j)-i)\left((n-d) \bmod 2+1\right)+i\right) \bmod 3 \\ &= \left(\left(\frac{\mu}{2^d} - \frac{1}{2}\right)(j-i)\left(((n+1)-d) \bmod 2+1\right)+i\right) \bmod 3 \end{aligned}$$

(using $\forall \kappa \in \mathbf{N}_0: 3 \mid 2^{2\kappa} - 1$),

$$j_\mu(p^{i,j;n+1}) = \ldots \text{ (analogously)}. \quad \square$$

ii) Parallel algorithm. A striking consequence of Proposition 1 is a formula which completely covers the shortest path.

PROPOSITION 2. *Let* $(i,j) \in \{0,1,2\}^2, i \neq j$. *Then for any* $\nu \in \{0, \ldots, 2^n - 1\}$ *and any* $d \in \{1, \ldots, n\}$:

$$p_\nu(d) := p_\nu^{i,j;n}(d) = \left((j-i)\,((n-d) \bmod 2+1)\,\mathrm{ent}\left(\frac{\nu}{2^d} + \frac{1}{2}\right) + i\right) \bmod 3 .$$

Proof. $p_0(d) = i$ and, by Proposition 1, for $\mu \in \{1, \ldots, 2^n - 1\}$:

$$(1) \quad p_\mu(d) = \begin{cases} p_{\mu-1}(d), & \text{if} \quad d \neq d_\mu(p); \\ (p_{\mu-1}(d) + (j-i)\,((n-d) \bmod 2+1)) \bmod 3, & \text{if} \quad d = d_\mu(p). \end{cases}$$

So $p_\nu(d) = ((j-i)\,((n-d) \bmod 2+1)\,|\{\mu \in \{1, \ldots, \nu\}; d = d_\mu(p)\}| + i) \bmod 3.$
But

$$(2) \qquad d = d_\mu(p) \Leftrightarrow \exists \kappa \in \mathbf{N}_0 : \mu = 2^{d-1} + \kappa 2^d ,$$

whence

$$|\{\mu \in \{1, \ldots, \nu\}; d = d_\mu(p)\}| = \min\{\lambda \in \mathbf{N}_0; \nu < 2^{d-1} + \lambda 2^d\}$$
$$= \mathrm{ent}\left(\frac{\nu}{2^d} + \frac{1}{2}\right). \quad \square$$

The observations from Proposition 1 contained in (1) and (2) can be used, in the special case $d = 1$, to yield the ultimate algorithm.

iii) Humane algorithm. The essence of the algorithm most suitable to a human being comes from the following statement, which is an immediate consequence of Proposition 1.

PROPOSITION 3. *In the shortest path from* $\hat{i}^n$ *to* $\hat{j}^n$ $((i,j) \in \{0,1,2\}^2, i \neq j)$, *disc* 1 *is moved in the* μ-*th move if and only if* μ *is odd. It then moves in cyclic order*

from i *through* j *to* $i \circ j$, *if* n *is odd;*

from i *through* $i \circ j$ *to* j, *if* n *is even.*

Following Proposition 3 for odd moves, even moves are dictated by rule (0), so that the shortest path can be carried out rather speedy.

It has become obvious that the shortest path between perfect states can be made very transparent. It is even possible, by an inversion of Proposition 2, to construct a fast (i.e. $O(n)$) algorithm which decides if a given state $r \in T_n$ appears in the shortest path from $\hat{i}^n$ to $\hat{j}^n$ and, if it does, gives the number μ of moves it took to reach it starting from $\hat{i}^n$. This allows to continue the solution abandoned at a certain stage by somebody. Similarly, one can also determine μ if one finds a person who has died with a disc in his hand carrying through the shortest path. If, however, someone has committed an error during the effectuation, it is necessary to know how to solve 𝔓1.

1.3. Problems 1 and 2

By Theorem 1, the existence of a shortest path from $s \in T_n$ to $t \in T_n$ is guaranteed.

Definition 5. Let $(s, t) \in T_n^2$. Then $\mu(s, t)$ denotes the length of the shortest path from s to t; if $t = \hat{j}^n$, it will be written $\mu(s; j)$.

In this section for any pair (s, t) of regular states $\mu(s, t)$ will be determined and the shortest path(s) constructed. Finally, average values of μ will be deduced.

1.3.0. Construction of the shortest paths between regular states

Although 𝔓*s* 1 and 2 have been considered in literature (see Introduction), there is no proof of minimality in any of these papers, since everybody assumed that in a shortest path the largest disc moves only once (if at all). Example 1 shows the wrongness of this assumption. However, the following is true.

Lemma 1. *Let $p \in P_{n+1}(s, t)$ be shortest. Then disc $n + 1$ moves*

o) *not at all if and only if $s(n+1) = t(n+1)$,*

i) *at most once if s or t is perfect,*

ii) *at most twice in general.*

Remark 3 and Definition 6. For $p \in P_n(s, t)$ define $-p \in T_n^{\mu_p+1}$ by

$$\forall \nu \in \{0, \ldots, \mu_p\} : -p_\nu = p_{\mu_p - \nu} .$$

It is easy to see that $-p \in P_n(t, s)$ and therefore it is clear that $-p$ is shortest iff p has this property. In view of this, part i of Lemma 1 will be proved for perfect t only.

Proof of Lemma 1. First observe that disc $n + 1$, once moved away from peg $k \in \{0, 1, 2\}$ during a shortest path p, will never come back to that peg, for suppose

$$\exists \mu', \mu'' \in \{1, \ldots, \mu_p\}, \ \mu' < \mu'' : d_{\mu'}(p) = d_{\mu''}(p) = n + 1 , \ i_{\mu'}(p) = j_{\mu''}(p) = k$$

and define a new path $\tilde{p}$ by deleting all the moves μ from p with $\mu' \leqslant \mu \leqslant \mu''$, $d_\mu(p) = n + 1$, then $\tilde{p} \in P_{n+1}(s, t)$ (the position of disc $n + 1$ does not limit the moves of the other discs!) and is shorter than p. This already proves o (the other part of o is trivial) and ii.

Now assume, for the proof of i, that disc $n + 1$ moves twice in a shortest path p, in moves μ' and $\mu''(1 \leqslant \mu' < \mu'' \leqslant \mu_p)$ say. Then necessarily $\bar{p}_{\mu'} = \widehat{t(n+1)^n}$, $\bar{p}_{\mu''} = \widehat{s(n+1)^n}$ and, as t is supposed to be perfect, $\bar{p}_{\mu_p} = \widehat{t(n+1)^n}$, But this implies, by Theorem 2, $\mu_p - \mu'' \geqslant 2^n - 1$ and $\mu'' - 1 - \mu' \geqslant 2^n - 1$, such that $\mu_p \geqslant 2^{n+1} - 1 + \mu' \geqslant 2^{n+1}$, contradicting Theorem 1. □

With Lemma 1 on hand, it is now easy to construct shortest paths between regular states. Although the solution of 𝔓2 contains of course the solution of 𝔓1, it is convenient to state and prove the cases separately. The following definition will be useful.

Definition 7. For $r \in T_n$ and $j \in \{0, 1, 2\}$ let $r^j: \{0, ..., n\} \to \{0, 1, 2\}$ be defined by

$$(3) \qquad \begin{cases} r^j(n) = j\,, \\ \forall 0 \leqslant d < n: r^j(d) = r^j(d+1) \circ r(d+1)\,. \end{cases}$$

Note that (3) $\Leftrightarrow \forall 0 \leqslant d \leqslant n: r^j(d) = \left((-1)^{n-d}\{j + \sum_{c=d+1}^{n} (-1)^{n-c} r(c)\}\right) \bmod 3.$

THEOREM 3. *Let* $r \in T_n$ *and* $j \in \{0, 1, 2\}$. *Then*

$$\mu(r;j) = \sum_{\substack{d \in \{1, ..., n\} \\ r(d) \neq r^j(d)}} 2^{d-1}\,;$$

the shortest path from r *to* $\hat{j}$ *is unique and can be constructed in the following way:*

Beginning with r, *do:* (*for* $d = 1$ *to* n: (if $r(d) \neq r^j(d)$: (*move disc* d *from* $r(d)$ *to* $r^j(d)$ *and do* $p^{r^j(d-1), r^j(d); d-1}$))).

Definition 8. The shortest path from r to $\hat{j}^n$ will be denoted by $p^{r;j}$.

Proof of Theorem 3 by induction on n.

a) For $n = 0$ the statement is trivial.

b) If $r(n+1) = j$, then by Lemma 1o disc $n + 1$ is not moved at all, and the shortest path from r to $\hat{j}^{n+1}$ is given by

$$\forall \nu \in \{0, ..., \mu(\bar{r};j)\}: \overline{p_\nu^{r;j}} = p_\nu^{\bar{r};j}, p_\nu^{r;j}(n+1) = j;$$

the statements of the theorem follow easily using (3).

If $r(n+1) \neq j$, let $k := j \circ r(n+1)$; then by Lemma 1o and i, disc $n+1$ is moved exactly once and so the shortest path from r to $\hat{j}^{n+1}$ is given by

$$\forall \nu \in \{0, ..., \mu(\bar{r};k)\}: \overline{p_\nu^{r;j}} = p_\nu^{\bar{r};k}, p_\nu^{r;j}(n+1) = r(n+1),$$

$$\forall \nu \in \{\mu(\bar{r};k) + 1, ..., \mu(\bar{r};k) + 2^n\}: \overline{p_\nu^{r;j}} = p_{\nu-\mu(\bar{r};k)-1}^{k,j;n}, p_\nu^{r;j}(n+1) = j,$$

from which again the statements of the theorem follow using (3). □

As an example, $\mu(r;0) = 164$ for the r of Figure 1.

For presenting the solution of $\mathfrak{P}2$ it is, of course, no loss of generality to disregard the case of an empty **TH** and, in view of Lemma 1o, to assume that the largest disc is on different pegs in s and t. The following definition is needed.

Definition 9. Let $(s,t) \in T_{n+1}^2$, $s(n+1) \neq t(n+1)$. Then

$$\mu_1(s,t) := 1 + \mu(\bar{s}; s(n+1)\circ t(n+1)) + \mu(\bar{t}; s(n+1)\circ t(n+1)),$$
$$\mu_2(s,t) := 2^n + 1 + \mu(\bar{s}; t(n+1)) + \mu(\bar{t}; s(n+1)).$$

THEOREM 4. *Let* $(s,t) \in T_{n+1}^2$, $s(n+1) \neq t(n+1)$. *Then* $\mu(s,t) = \min\{\mu_1(s,t), \mu_2(s,t)\}$. *There are exactly two shortest paths from* s *to* t *if* $\mu_1(s,t) = \mu_2(s,t)$, *otherwise the shortest path is unique. The shortest path(s) can be constructed thus:*

if $\mu = \mu_1$: *Beginning with* s, *do* $p^{\bar{s};s(n+1)\circ t(n+1)}$, *move disc* $n+1$ *from* $s(n+1)$ *to* $t(n+1)$, *do* $-p^{\bar{t};s(n+1)\circ t(n+1)}$;

if $\mu = \mu_2$: *Beginning with* s, *do* $p^{\bar{s};t(n+1)}$, *move disc* $n+1$ *from* $s(n+1)$ *to* $s(n+1)\circ t(n+1)$, *do* $p^{t(n+1),s(n+1);n}$, *move disc* $n+1$ *from* $s(n+1)\circ t(n+1)$ *to* $t(n+1)$, *do* $-p^{\bar{t};s(n+1)}$.

Proof. It follows immediately from Lemma 1ii and Theorem 3 that the paths described in the statement of Theorem 4 are the only candidates for a shortest path from s to t. So one just has to choose the shorter of the two or both if their length is equal. □

Remark 4. It is easy to see that, using Theorems 4 and 3, it is possible to reduce $\mathfrak{P}2$ to the solution of $\mathfrak{P}0$, so that any of the algorithms in 1.2.1 can be employed to construct an algorithm for the solution of $\mathfrak{P}2$.

Although for any $(s, t) \in T_n^2$ the length $\mu(s, t)$ of the shortest path(s) from s to t can easily be calculated now, it is nevertheless interesting to know the average length of shortest paths explicitely. This will be examined in the following two subsections.

1.3.1. DISCUSSION OF THE MINIMAL LENGTH $\mu(r;j)$

A short glance at the graph of the **TH** (Figure 2) suggests the following results.

PROPOSITION 4. *Let* $j \in \{0, 1, 2\}$. *Then* $\gamma_n := \sum_{r\in T_n} \mu(r;j) = 3^n \cdot \frac{2}{3}(2^n - 1)$.

COROLLARY 1. *The average length of shortest paths from regular to perfect states is* 2/3 *of the maximal length.*

The corollary follows immediately from Proposition 4, together with Theorems 0 and 2.

Proof of Proposition 4. $\gamma_0 = 0$ and Theorem 3 yields

$$\forall n \in \mathbf{N}_0 : \gamma_{n+1} = \sum_{\substack{r\in T_{n+1}\\ r(n+1)=j}} \mu(r;j) + \sum_{\substack{r\in T_{n+1}\\ r(n+1)\neq j}} \mu(r;j)$$

$$= \gamma_n + 2 \cdot 3^n \left(\frac{\gamma_n}{3^n} + 2^n\right) = 3\gamma_n + 2 \cdot 6^n .$$

Thus $\gamma_n = 2 \sum_{\kappa=0}^{n-1} 3^\kappa 6^{n-1-\kappa} = \frac{2}{3}(6^n - 3^n)$, where use has been made of

(4) $$\forall a \in \mathbf{R} \forall (a_n), (\alpha_n) \in \mathbf{R}^{\mathbf{N}_0} : ((\alpha_0 = 0 \wedge \forall n \in \mathbf{N}_0 : \alpha_{n+1} = a\alpha_n + a_n)$$
$$\Leftrightarrow (\forall n \in \mathbf{N}_0 : \alpha_n = \sum_{\kappa=0}^{n-1} a^\kappa a_{n-1-\kappa}))$$

and

(5) $$\forall (a, b) \in \mathbf{R}^2, a \neq b \ \forall n \in \mathbf{N}_0 : \sum_{\kappa=0}^{n-1} b^\kappa a^{n-1-\kappa} = \frac{a^n - b^n}{a - b}. \qquad \square$$

The following is an interesting observation.

PROPOSITION 5. *Let* $\mu \in \{0, \ldots, 2^n - 1\}$. *Then* $|\{r \in T_n ; \mu(r;j) = \mu\}| = 2^{\beta(\mu)}$, *where* $\beta(\mu)$ *is the number of non-zero binary digits of* μ.

Remark 5. This is the population number of the μ-th level in the shortest path tree for $\hat{j}^n$, constructed for example for $j = 0$ (and $n = 3$) from Figure 2 by deleting all horizontal edges.

Proposition 5 is an easy consequence of the formula for $\mu(r;j)$ in Theorem 3 in view of (3). It can also serve as the base of an alternative proof of Proposition 4; this idea will be useful in the following subsection.

1.3.2. Discussion of the minimal length $\mu(s, t)$

The function $\mu(s, t)$ is much more puzzling than $\mu(r;j)$ because of the decision between μ_1 and μ_2 in Theorem 4. Although there seems to be no handy method, other than sheer computation, to find out, for given $(s, t) \in T^2_{n+1}$, which of the two is smaller, one can determine the number of events for each case.

Proposition 6.

i) $$|\{(s, t) \in T^2_{n+1};\, s(n+1) \neq t(n+1),\, \mu_1(s, t) = \mu_2(s, t)\}| = \frac{6}{\sqrt{17}}(\Theta^n_+ - \Theta^n_-),$$

ii) $$|\{(s, t) \in T^2_{n+1};\, s(n+1) \neq t(n+1),\, \mu_1(s, t) > \mu_2(s, t)\}|$$
$$= \frac{3}{7}9^n - \frac{3}{7}2^n - \frac{3}{\sqrt{17}}(\Theta^n_+ - \Theta^n_-),$$

iii) $$|\{(s, t) \in T^2_{n+1};\, s(n+1) \neq t(n+1),\, \mu_1(s, t) < \mu_2(s, t)\}|$$
$$= \frac{39}{7}9^n + \frac{3}{7}2^n - \frac{3}{\sqrt{17}}(\Theta^n_+ - \Theta^n_-);$$

here $\Theta_\pm := \frac{1}{2}(5 \pm \sqrt{17})$.

Remark 6. This is the first time, an irrational number enters, though implicitely, into considerations about the **TH**! By the way, $\sqrt{17}$ is one of the "oldest" irrationals, a proof for its incommensurability with unity being known in $-$ 398 to Theodorus of Cyrene (cp. [47, p. 141 ff]).

Corollary 2. *Asymptotically (for large n), the largest disc moves*

o) *not at all in* $\frac{1}{3}$,

i) *exactly once in* $\frac{13}{21}$,

ii) *exactly twice in* $\frac{1}{21}$

of all shortest paths between regular states.

This is an immediate consequence of Proposition 6 and the construction of the shortest path in Theorem 4.

The following functions will be useful in the proof of Proposition 6.

Definition 10. Let $\forall \mu \in \mathbf{Z}: z_n(\mu) = |\{r \in T_n; \mu(r;i) - \mu(r;j) = \mu\}|$; here (i, j) is any pair of distinct elements of $\{0, 1, 2\}$, and it is clear by symmetry that the definition does not depend on the specific pair employed.

The following lemma is a summary of properties of these functions.

LEMMA 2. o) $z_0(0) = 1, \forall \mu \in \mathbf{Z}\backslash\{0\}: z_0(\mu) = 0,$

$$\forall n \in \mathbf{N}_0\ \forall \mu \in \mathbf{Z}: z_{n+1}(\mu) = z_n(\mu-2^n) + z_n(\mu) + z_n(\mu+2^n);$$

i) $\forall \mu \in \mathbf{Z}: z_n(-\mu) = z_n(\mu), z_n(0) = 1, z_n(1) = n, z_n(2^n-1) = 1,$

$|\mu| \geqslant 2^n \Rightarrow z_n(\mu) = 0;$

ii) $\sum_{\mu \in \mathbf{Z}} z_n(\mu) = 3^n, \sum_{\mu \in \mathbf{N}} z_n(\mu) = \frac{1}{2}(3^n-1), \sum_{\mu \in \mathbf{N}} \mu z_n(\mu) = \frac{1}{5}(6^n-1);$

iii) *let* $x_n := \sum_{\mu \in \mathbf{N}} z_n(\mu)\, z_n(2^n-\mu), y_n := \sum_{\mu \in \mathbf{N}} z_n^2(\mu),$ *then*

$$x_n = \frac{1}{\sqrt{17}}(\Theta_+^n - \Theta_-^n),\ y_n = \frac{1}{4}\left(\left(1 + \frac{1}{\sqrt{17}}\right)\Theta_+^n + \left(1 - \frac{1}{\sqrt{17}}\right)\Theta_-^n - 2\right).$$

Proof. o) The statements about z_0 are trivial. The recursion relation is obtained from the fact

$$\mu(r;i) - \mu(r;j) = \begin{cases} \mu(\bar{r};i) - \mu(\bar{r};i\circ j) - 2^n, & \text{if} \quad r(n+1) = i, \\ \mu(\bar{r};i\circ j) - \mu(\bar{r};j) + 2^n, & \text{if} \quad r(n+1) = j, \\ \mu(\bar{r};j) - \mu(\bar{r};i), & \text{if} \quad r(n+1) = i \circ j, \end{cases}$$

which in turn follows from the construction in the proof of Theorem 3.

i) is proved by induction on n using o.

ii) is proved by induction on n using o and i.

iii) By o and i: $x_0 = 0, y_0 = 0, y_1 = 1$ and

$$\forall n \in \mathbf{N}_0: x_{n+1} = 2x_n + 2y_n + 1,\ y_{n+1} = 2x_n + 3y_n + 1,$$

such that $x_{n+1} = y_{n+1} - y_n$ and $y_{n+2} = 5y_{n+1} - 2y_n + 1$.

Defining $\eta_n := y_n + \frac{1}{2}$, the following recurrent sequence has to be calculated:

$$(6) \qquad \begin{cases} \eta_0 = \frac{1}{2}, \eta_1 = \frac{3}{2}, \\ \forall n \in \mathbf{N}_0 : \eta_{n+2} = 5\eta_{n+1} - 2\eta_n . \end{cases}$$

The ansatz $\tilde{\eta}_n = \Theta^n$ with a $\Theta \in \mathbf{R}$ leads to the solutions $\tilde{\eta}_n = \Theta_\pm^n$ of the recurrence relation, such that $\eta_n = \frac{1}{4}\left(\left(1 + \frac{1}{\sqrt{17}}\right)\Theta_+^n + \left(1 - \frac{1}{\sqrt{17}}\right)\Theta_-^n\right)$. The formulas for x_n and y_n are obtained from this by simple calculations. □

Proof of Proposition 6. i) Let $(s, t) \in T_{n+1}^2$, $s(n+1) \neq t(n+1)$, and define

$$\mu := \mu(\bar{s}; s(n+1) \circ t(n+1)) - \mu(\bar{s}; t(n+1)),$$
$$\tilde{\mu} := \mu(\bar{t}; s(n+1) \circ t(n+1)) - \mu(\bar{t}; s(n+1)).$$

Then $\mu_1(s, t) - \mu_2(s, t) = \mu + \tilde{\mu} - 2^n$ and

$$\mu_1(s, t) = \mu_2(s, t) \Leftrightarrow \mu, \tilde{\mu} \in \{1, \ldots, 2^n - 1\}, \tilde{\mu} = 2^n - \mu .$$

Thus, in view of the six different choices for $(s(n+1), t(n+1))$,

$$|\{(s, t) \in T_{n+1}^2 ; s(n+1) \neq t(n+1), \mu_1(s, t) = \mu_2(s, t)\}| = 6x_n ,$$

and Lemma 2 completes the proof of i.

ii) By a similar argument and with $\nu = 2^n - \tilde{\mu}$:

$$|\{(s, t) \in T_{n+1}^2 ; s(n+1) \neq t(n+1), \mu_1(s, t) > \mu_2(s, t)\}| = 6w_n ,$$

where $w_n := \sum_{\mu \in \mathbf{N}} \sum_{\nu=1}^{\mu-1} z_n(\mu) z_n(2^n - \nu)$. It is easy to see, using Lemma 2, that $w_0 = 0$ and $n \in \mathbf{N}_0 : w_{n+1} = 2w_n - y_n + \frac{1}{2}(3^{2n} - 1)$, which yields, by (4) and (5), the desired result.

iii) follows from

$$\begin{aligned} 3^{2(n+1)} &= |\{(s, t) \in T_{n+1}^2 ; s(n+1) = t(n+1)\}| \\ &+ |\{(s, t) \in T_{n+1}^2 ; s(n+1) \neq t(n+1), \mu_1(s, t) < \mu_2(s, t)\}| \\ &+ |\{(s, t) \in T_{n+1}^2 ; s(n+1) \neq t(n+1), \mu_1(s, t) > \mu_2(s, t)\}| \\ &+ |\{(s, t) \in T_{n+1}^2 ; s(n+1) \neq t(n+1), \mu_1(s, t) = \mu_2(s, t)\}| . \end{aligned}$$ □

By the same methods, the total and average number of moves in shortest paths between all regular states can be determined now.

PROPOSITION 7.

$$\delta_n := \sum_{(s,t)\in T_n^2} \mu(s,t) = \frac{466}{885}18^n - \frac{1}{3}9^n - \frac{3}{5}3^n + \left(\frac{12}{59} + \frac{18}{1003}\sqrt{17}\right)\Theta_+^n + \left(\frac{12}{59} - \frac{18}{1003}\sqrt{17}\right)\Theta_-^n .$$

COROLLARY 3. *Asymptotically (for large n) the average length of shortest paths between regular states is* $\frac{466}{885}$ *of the maximal length.*

Again, this is an immediate consequence of Proposition 7 by Theorems 0 and 2.

Proof of Proposition 7. Clearly, $\delta_0 = 0$; let $n \in \mathbf{N}_0$; then

$$(7)\quad \begin{aligned}\delta_{n+1} &= \sum_{\substack{(s,t)\in T_{n+1}^2\\ s(n+1)=t(n+1)}} \mu(s,t) + \sum_{\substack{(s,t)\in T_{n+1}^2\\ s(n+1)\neq t(n+1)}} \mu(s,t)\\ &= 3\delta_n + \sum_{\substack{(s,t)\in T_{n+1}^2\\ s(n+1)\neq t(n+1)}} \mu_1(s,t) - \sum_{\substack{(s,t)\in T_{n+1}^2\\ s(n+1)\neq t(n+1)\\ \mu_1(s,t)>\mu_2(s,t)}} \big(\mu_1(s,t)-\mu_2(s,t)\big).\end{aligned}$$

Let $(i,j) \in \{0, 1, 2\}^2$, $i \neq j$. Then

$$(8)\quad \begin{aligned}\sum_{\substack{(s,t)\in T_{n+1}^2\\ s(n+1)\neq t(n+1)}} \mu_1(s,t) &= 6 \sum_{\substack{(s,t)\in T_{n+1}^2\\ s(n+1)=i\\ t(n+1)=j}} \mu_1(s,t) = 6\cdot 3^{2n}\left(\frac{\gamma_n}{3^n} + 1 + \frac{\gamma_n}{3^n}\right)\\ &= 2\cdot 3^{2n}(2^{n+2}-1).\end{aligned}$$

Using the same arguments as in the proof of Proposition 6, one gets

$$\sum_{\substack{(s,t)\in T_{n+1}^2\\ s(n+1)\neq t(n+1)\\ \mu_1(s,t)>\mu_2(s,t)}} \big(\mu_1(s,t)-\mu_2(s,t)\big) = 6u_n, \quad \text{where}$$

$$u_n := \sum_{\mu\in\mathbf{N}} \sum_{\nu=1}^{\mu-1} (\mu-\nu)z_n(\mu)z_n(2^n-\nu).$$

To calculate u_n, the following must be defined:

$$v_n := \sum_{\mu\in\mathbf{N}} \sum_{\nu=1}^{\mu-1} (\mu-\nu)z_n(\mu)z_n(\nu).$$

Then the recursion relation holds:

$$(9)\quad \begin{cases} u_0 = v_0 = 0, \\ \forall n \in \mathbf{N}_0 : u_{n+1} = 2u_n + 2v_n + \frac{1}{5}(3^n+1)(6^n-1), \\ v_{n+1} = 2u_n + 3v_n + \frac{1}{5}(6^n-1) + \frac{1}{2}6^n(3^n-1); \end{cases}$$

this is proved with the aid of Lemma 2 and the facts

$$\forall n \in \mathbf{N}_0 : \sum_{\mu=1}^{2^n-1} \sum_{\nu=1}^{\mu-1} (\mu-\nu) z_n(2^n-\mu) z_n(2^n-\nu) = v_n,$$

$$\sum_{\mu=1}^{2^n-1} \sum_{\nu=1}^{\mu-1} (\mu-\nu) z_n(2^n-\mu) z_n(\nu) = u_n + 2^{n-2}(3^n-1)^2 - \frac{1}{5}(3^n-1)(6^n-1),$$

which in turn follow from Lemma 2.

The solution of (9) is (analogously to (4))

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \sum_{\kappa=0}^{n-1} A^\kappa \begin{pmatrix} a_{n-1-\kappa} \\ b_{n-1-\kappa} \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix},$$

$$\forall \lambda \in \mathbf{N}_0 : a_\lambda = \frac{1}{5}(3^\lambda+1)(6^\lambda-1),$$

$$b_\lambda = \frac{1}{5}(6^\lambda-1) + \frac{1}{2}6^\lambda(3^\lambda-1).$$

Defining $\forall \kappa \in \mathbf{N}_0 : \eta_\kappa := \frac{1}{2}(A^\kappa)_{1,1} + \frac{1}{4}(A^\kappa)_{1,2}$, it turns out that $(A^{\kappa+1})_{1,2} = 2(\eta_{\kappa+1} - \eta_\kappa)$ and that $(\eta_\kappa)_{\kappa \in \mathbf{N}_0}$ fulfils (6). Thus $(A^\kappa)_{1,2} = \frac{2}{\sqrt{17}}(\Theta_+^\kappa - \Theta_-^\kappa)$ and $(A^\kappa)_{1,1} = \frac{1}{2}\left(\left(1 - \frac{1}{\sqrt{17}}\right)\Theta_+^\kappa + \left(1 + \frac{1}{\sqrt{17}}\right)\Theta_-^\kappa\right)$. A careful computation, with the aid of (5), yields

$$\forall n \in \mathbf{N}_0 : u_n = \frac{1}{59}18^n - \left(\frac{1}{118} + \frac{31}{2006}\sqrt{17}\right)\Theta_+^n - \left(\frac{1}{118} - \frac{31}{2006}\sqrt{17}\right)\Theta_-^n.$$

Inserting this and (8) into (7) leads to

$$\forall n \in \mathbf{N}_0 : \delta_{n+1} = 3\delta_n + \frac{466}{59}18^n - 2\cdot 9^n + \left(\frac{3}{59} + \frac{93}{1003}\sqrt{17}\right)\Theta_+^n + \left(\frac{3}{59} - \frac{93}{1003}\sqrt{17}\right)\Theta_-^n,$$

and again with (4) and (5) the formula for δ_n is established. □

2. Irregular states

Although introduced already in the leaflet [8] to the original **TH** puzzle, Lucas's second problem (cp. also [9]) has not yet received an adequate mathematical treatment. The reason is that the violation of the regularity assumption on the initial state takes away a great deal of symmetry from the considerations. In particular, the mathematical model has to be changed.

2.0. Mathematical model

With pegs 0, 1, 2 set up from left to right and counting positions of discs and bottoms of pegs from top left to bottom right in a given state, one can attach to each *disc* $d \in \{1, ..., n\}$ its position $\rho(d)$ in this enumeration, and to the *bottom of peg* $i \in \{0, 1, 2\}$ its position $\rho(n+1+i)$. This leads to the following definition.

Definition 11.

$$\mathfrak{T}_n := \{\rho : \{1, ..., n+3\} \overset{1-1}{\underset{\text{onto}}{\rightarrow}} \{1, ..., n+3\}\,;\ \rho(n+1) < \rho(n+2) < \rho(n+3) \doteq n+3\}\,.$$

As any $\rho \in \mathfrak{T}_n$ corresponds to a state of the **TH**, it follows immediately:

Theorem 5. *The number of states of the* **TH** *with* n *discs is* $\dfrac{(n+2)!}{2}$.

Remark 7. Surprisingly, Lucas writes that for $n = 64$, this number has "more than fifty figures" (see [8]); although this is true, it falls short by some fourty powers of ten!

While the description of a state is simple, the rules of a move are clumsy in this model and far from intuition. So it is convenient to construct the following imbedding.

Definition 12.

$$J : \mathfrak{T}_n \overset{1-1}{\rightarrow} \{(r, h)\,;\ r : \{1, ..., n+3\} \rightarrow \{0, 1, 2\},\ h : \{1, ..., n+3\} \rightarrow \{0, ..., n\}\}\,,$$

$$\rho \mapsto (r, h)\,,$$

where

$$\forall d \in \{1, ..., n+3\}:$$

$$h(d) = \min\{\rho(n+1+i) - \rho(d)\,;\ \rho(n+1+i) \geqslant \rho(d),\ i \in \{0, 1, 2\}\}\,,$$

and $r(d)$ is the i for which this minimum is attained.

It is easily checked that J is an injection, and so $\mathfrak{T}_n$ and $J\mathfrak{T}_n$ will be identified, i.e. ρ and (r, h) will be used interchangeably. Furthermore, as $r(d)$ and $h(d)$ do not depend on ρ for $d = n + 1 + i$, $i \in \{0, 1, 2\}$, r will be identified with $r \mid \{1, ..., n\} \in T_n$ and h with $h \mid \{1, ..., n\}$. $\rho \in \mathfrak{T}_n$ will also be written $[(r(1), h(1)), ..., (r(n), h(n))]$. Again, $r(d)$ is the peg onto which disc d is stacked and $h(d)$ is its level above the bottom of that peg. In addition, by $r \mapsto (r, h)$ with $\forall d \in \{1, ..., n\}: h(d) = |\{c \in \{d, ..., n\}; r(c) = r(d)\}|$ an injection is given from T_n into $\mathfrak{T}_n$ and again r and (r, h) will be identified.

Definition 13. A pair $(\rho_0, \rho_1) \in \mathfrak{T}_n^2$ is a *(legal) move* (of disc d from peg i to peg j), iff

$$\exists (i, j) \in \{0, 1, 2\}^2, i \neq j: d := \operatorname{top}(\rho_0; i) < \min\{n+1, \operatorname{top}(\rho_0; j)\}$$
$$\wedge \; \big(r_1(d) = j,\ h_1(d) = h_0(\operatorname{top}(\rho_0; j)) + 1,\ \forall c \in \{1, ..., n\} \setminus \{d\}: r_1(c) = r_0(c),$$
$$h_1(c) = h_0(c)\big),$$

where

$$\forall \rho \in \mathfrak{T}_n\ \forall i \in \{0, 1, 2\}: \operatorname{top}(\rho; i) \in \{1, ..., n+3\}$$

with

$$r(\operatorname{top}(\rho; i)) = i,\ h(\operatorname{top}(\rho; i)) = \max h(r^{-1}(\{i\})).$$

For $(\sigma, \tau) \in \mathfrak{T}_n^2$, a *path* $\pi \in \Pi_n(\sigma, \tau)$ from σ to τ and its *length* are defined as in Chapter 1.

Remark 8. If ρ_0 is regular in a move $(\rho_0, \rho_1) \in \mathfrak{T}_n^2$, then so is ρ_1, and (ρ_0, ρ_1) is a legal move in the sense of Definition 1. As the same applies to paths, it is clear that no new paths between regular states turn up.

The analogue to Definition 2i is

Definition 14. For any $\rho \in \mathfrak{T}_n$ and $d \in \{1, ..., n+3\}$ let

$$U_\rho^d := \{c \in \{1, ..., n\}; r(c) = r(d) \wedge h(c) \leqslant h(d)\}$$

and define $\bar{\rho}^d \in \mathfrak{T}_{n-h(d)}$ by

$$\forall c \in \{1, ..., n-h(d)\}: \bar{r}^d(c) = r(\iota(c)),$$
$$\bar{h}^d(c) = \begin{cases} h(\iota(c)) - h(d), & \text{if } \; r(\iota(c)) = r(d), \\ h(\iota(c)), & \text{else}, \end{cases}$$

where $\iota: \{1, ..., n-h(d)\} \to \complement U_\rho^d$ is strictly increasing; if $d = n$, $\bar{\rho}^d$ will be written $\bar{\rho}$ simply.

Similarly, $\underline{\rho_d} \in \mathfrak{T}_{h(d)}$ is defined by

$$\forall c \in \{1, ..., h(d)\} : \underline{r_d}(c) = r(d), \underline{h_d}(c) = h(\iota(c)),$$

where now $\iota: \{1, ..., h(d)\} \to U_\rho^d$ is strictly increasing.

Remark 9. Given U_ρ^d, it is possible to reconstruct ρ from $\underline{\rho_d}$ and $\bar{\rho}^d$. Thus, as long as disc d does not move, any move $(\bar{\rho}_0, \bar{\rho}_1) \in \mathfrak{T}_{n-h(d)}^2$ is equivalent to a move $(\rho_0, \rho_1) \in \mathfrak{T}_n^2$, provided that $d > \max \complement U_\rho^d$. This will frequently be used in the sequel.

2.1. Existence of a shortest path from a state to a regular state and an upper bound for its length

In contrast to the situation for regular states, $(\rho_1, \rho_0) \in \mathfrak{T}_n^2$ is not necessarily a legal move if (ρ_0, ρ_1) is. So one can not expect $\Pi_n(\sigma, \tau)$ to be non-empty for every pair $(\sigma, \tau) \in \mathfrak{T}_n^2$. The goal of this section will be:

THEOREM 6. *Let $n \in \mathbf{N} \setminus \{1\}$. For any pair $(\sigma, t) \in \mathfrak{T}_n \times T_n$ there is a (shortest) path from σ to t with length less than or equal to $2^n - 1 + 2^{n-2}$.*

Remark 10. i) The restriction on n is not serious, since there are no irregular states for $n \in \{0, 1\}$.

ii) The bound on the length of a shortest path in Theorem 6 is sharp:

Example 2. $\sigma = [(0, n), (0, n-1), ..., (0, 3), (0, 1), (0, 2)]$, $t = \hat{0}^n$. Before the first move of disc n (it has to be moved to arrive at a regular state!), to peg 1 for instance, discs 1 to $n - 2$, which are regularly distributed on top of it, have to be moved to peg 2. So, by Theorem 2, at least 2^{n-2} moves have been carried out after the first move of disc n, when a regular state is reached from which it takes another $2^n - 1$ moves to arrive at t, as can be calculated using Theorem 3.

To prove Theorem 6, some preparations have to be done.

LEMMA 3. *For every $\rho \in \mathfrak{T}_n$ and $j \in \{0, 1, 2\}$ there is a $\tilde{\rho} \in \mathfrak{T}_n$ with $\tilde{r} = \hat{j}^n$ and a path from ρ to $\tilde{\rho}$ with length less than or equal to $2^n - 1$; if $n = 0$ or $r(n) \neq j$, then $\tilde{\rho}$ is regular.*

Proof by induction on n. a) The case $n = 0$ is trivial.

b) If $r(n+1) = j$, then the induction hypothesis can be applied to $\bar{\rho}$, resulting in a $\tilde{\rho}$ and a path from ρ to $\tilde{\rho}$ in the spirit of Remark 9.

If $r(n+1) \neq j$, then transfer $\bar{\rho}$ to $j \circ r(n+1)$, which takes at most $2^n - 1$ moves by hypothesis, move disc $n + 1$ to j and then the first n discs to j, together at most $2^{n+1} - 1$ moves. As in the last action (if $n \neq 0$) disc n started from a peg different from j, the resulting state is regular by hypothesis. □

This lemma leads to the following interesting result:

PROPOSITION 8. *Let* $n \in \mathbf{N}\backslash\{1\}$. *For any* $\sigma \in \mathfrak{T}_n$ *there is a* $\tilde{t} \in T_n$ *and a path from* σ *to* $\tilde{t}$ *with length less than or equal to* 2^{n-2}.

Remark 11. Here again Example 2 shows that the bound on the length is sharp: Suppose for the σ of Example 2 there is a $\tilde{t} \in T_n$ and a path from σ to $\tilde{t}$ of length less than 2^{n-2}; then by Theorem 1, there is a path from σ to $t = \hat{0}$ of length less than $2^n - 1 + 2^{n-2}$, which contradicts the discussion of Example 2.

Proof of Proposition 8 by induction on n. a) For $n = 2$, the only irregular states are $[(j, 1), (j, 2)]$ for $j \in \{0, 1, 2\}$. Here it suffices to move disc 2 to a different peg to get a regular state.

b) If $h(n+1) = 1$, then the induction hypothesis can be applied to $\bar{\sigma}$. Otherwise, the transfer of $\bar{\sigma}$ to a peg j different from $s(n+1)$ and $s(n+1-h(n+1))$ is achieved in at most $2^{n+1-h(n+1)} - 1$ moves by Lemma 3. Then move disc $n + 1$ to $j \circ s(n+1)$. If $h(n+1) = 2$, the resulting state is regular and the number of moves at most 2^{n-1}. Otherwise, discs 1 to n can be transferred to a regular state in at most 2^{n-2} moves by hypothesis, and the total number of moves is less than or equal to $2^{n+1-h(n+1)} + 2^{n-2} \leqslant 2^{n-1}$. □

Now the proof of Theorem 6 is a trivial combination of Proposition 8 and Theorem 1.

Although Example 2 shows that shortest paths may be as long as $2^n - 1 + 2^{n-2}$, this worst case will not occur very frequently, as the following proposition tells, which will also be important in the subsequent sections.

PROPOSITION 9. *Let* $n \in \mathbf{N}\backslash\{1\}, (\sigma, t) \in \mathfrak{T}_n \times T_n$. *Then*

$$\mu(\sigma, t) \geqslant 2^n \Rightarrow (s(n) = t(n) \wedge h(n) > 1).$$

Proof. The proof is by constructing paths from σ to t shorter than 2^n for all cases different from the r.h.s. For convenience suppose that $n \geqslant 3$ (for $n = 2$, cp. the proof of Proposition 8).

i) $s(n) = t(n) \wedge h(n) = 1$. Then bring $\bar{\sigma}$ to $\bar{t}$, which takes at most $2^{n-1} - 1 + 2^{n-3}$ moves by Theorem 6.

ii) $s(n) \neq t(n) \wedge (h(n) > 1 \vee s(n-1) = s(n) \circ t(n))$. Then bring $\bar{\sigma}$ to peg $s(n) \circ t(n)$ in at most $2^{n-2} - 1$ moves (by Lemma 3), move disc n to $t(n)$ and then the other discs to $\bar{t}$ in at most another $2^{n-1} - 1 + 2^{n-3}$ moves by Theorem 6.

iii) $s(n) \neq t(n) \wedge (h(n) = 1 \wedge s(n-1) \neq s(n) \circ t(n))$. Then move $\bar{\sigma}$ to $\widehat{s(n) \circ t(n)}^n$ in at most $2^{n-1} - 1$ moves (by Lemma 3), move disc n to $t(n)$ and finally $\widehat{s(n) \circ t(n)}^n$ to $\bar{t}$ in at most $2^{n-1} - 1$ moves by Theorem 1. □

Remark 12. As in Theorem 1, the proof of Theorem 6 (Proposition 9) is constructive, allowing (if $s(n) \neq t(n) \vee h(n) = 1$) to find a path from σ to t with at most $2^n - 1 + 2^{n-2}$ $(2^n - 1)$ moves. But again, it does not necessarily lead to a shortest path, even if the steps are carried out efficiently; see Example 3 below. So the construction of shortest paths has to be discussed further.

2.2. CONSTRUCTION OF SHORTEST PATHS FROM A STATE TO A REGULAR STATE

Although it is now possible, in principle, to find all shortest paths from a state σ to a regular state t by sheer listing the paths between them not longer than the upper bound in Theorem 6, this crude proceeding is neither efficient nor does it provide any a priori information about the number of shortest paths. The following three lemmas will help to overcome these weaknesses.

LEMMA 4. *Let* $\pi \in \Pi_{n+1}(\sigma, t)$ *be shortest. Then disc* $n + 1$ *does not move twice to the same peg; consequently, it moves at most three times.*

Proof. Suppose $j \in \{0, 1, 2\}$ appears as goal of disc $n + 1$ at least twice in π, in moves μ' and $\mu''(\mu' < \mu'')$ say. Then, as $h_\mu(n+1) = 1$ after the first move of $n+1$, one can leave out all the moves μ with $d_\mu(\pi) = n+1$ and $\mu' < \mu \leqslant \mu''$ and gets a shorter path from σ to t. □

LEMMA 5. *Let* $j \in \{0, 1, 2\}, (\sigma, \tau) \in \mathfrak{T}_n^2$ *with* $t = \hat{j}^n$. *Then*

$$\Pi_n(\sigma, \tau) \neq \emptyset \Leftrightarrow \exists d \in \{1, ..., n+3\}, h_\sigma(d) = n$$
$$\vee\ d > \max \complement U_\sigma^d : U_\tau^d = U_\sigma^d \wedge \underline{\tau_d} = \underline{\sigma_d} \wedge \bar{\tau}^d = \hat{j}^{n-h_\sigma(d)}.$$

Proof. "⇒": If τ is regular, then take $d > n$. Otherwise

$$\{d \in \{1, ..., n\} ; \exists c \in \{1, ..., d-1\} : h_\tau(c) = h_\tau(d) - 1\} \neq \emptyset .$$

Choose the d with $h_\tau(d)$ a maximum. Then $h_\tau(d) = n$ or $d > \max \complement U_\tau^d$, and $\bar{\tau}^d = \hat{j}^{n-h_\tau(d)}$. Furthermore, as there is a path from σ to τ, $U_\sigma^d = U_\tau^d$ and $\underline{\sigma_d} = \underline{\tau_d}$, and so also $h_\sigma(d) = h_\tau(d)$.

"⇐" follows from Theorem 6. If τ is not regular, a path from σ to τ is given by a path from $\bar{\sigma}^d$ to $\bar{\tau}^d$ fixing disc d and the discs under it. □

LEMMA 6. *Let* $j \in \{0, 1, 2\}$.

i) *Let* $\sigma \in \mathfrak{T}_n, \tau_1, \tau_2$ *as in Lemma 5 for* d_1, d_2 *with* $h(d_1) \geqslant h(d_2)$. *Then* $\mu(\sigma, \tau_1) \leqslant \mu(\sigma, \tau_2)$.

ii) *Let* $i, k \in \{0, 1, 2\}, i \neq k, (\sigma_1, \sigma_2) \in \mathfrak{T}_n^2$ *with*

$$\forall d \in \{1, ..., n\} : (s_1(d) = s_2(d) =: s(d) \wedge s(d) \neq k \wedge (s(d) = i \Rightarrow h_1(d) = h_2(d))$$
$$\wedge\ (s(d) = i \circ k \Rightarrow h_2(d) = |\{c \in \{d, ..., n\} ; s(c) = i \circ k\}|)) ;$$

let $\tau_\kappa (\kappa \in \{1, 2\})$ *be as in Lemma 5 applied to* σ_κ *and* d_κ *with* $h_\kappa(d_\kappa)$ *a maximum.*

Then $\mu(\sigma_1, \tau_1) \leqslant \mu(\sigma_2, \tau_2)$.

Proof. i) Take a shortest path from σ to τ_2 and skip the moves of discs in $U_\sigma^{d_1}$.

ii) By induction on n. a) The case $n = 0$ is trivial.

b) By part i, it suffices to construct a path π_1 from σ_1 to peg j not longer than π_2, a given shortest path from σ_2 to τ_2.

The first, and possibly only, part of π_2 is equivalent to a path from $\bar{\sigma}_2$ to some peg $\tilde{j} \in \{0, 1, 2\}$. Define $\tilde{\sigma}_1 \in \mathfrak{T}_{n+1-h_2(n+1)}$ by deleting discs in $U_{\sigma_2}^{n+1}$ from σ_1 analogously to Definition 14. Then, by induction, there is a path from $\tilde{\sigma}_1$ to peg $\tilde{j}$ not longer than the former and by deleting all the moves of discs in $U_{\sigma_1}^{n+1}$ one gets a path $\tilde{\pi}_1$ from $\bar{\sigma}_1$ to peg $\tilde{j}$. If $s(n+1) = j$, then disc $n+1$ does not move in π_2, whence $\tilde{j} = j$ and

$\pi_1 = \tilde{\pi}_1$ does the job. Otherwise, add to $\tilde{\pi}_1$ the move, also present in π_2, of disc $n+1$ from $s(n+1)$ to $\tilde{j} \circ s(n+1)$. Now, if $s(n+1) = i \circ k$, a perfect n-state (perfect substate if $\tilde{j} = i$) moves from $\tilde{j}$ to some other peg in π_2, while in π_1 the latter peg can be reached in at most as many moves by Theorem 2 or Proposition 9. After that, or if $s(n+1) = i$, the induction hypothesis provides the rest of path π_1. □

By Lemma 4, the possible patterns of movements of the largest disc $n+1$ in a shortest path are determined, while Lemma 5 limits the number of cases to be considered before each move of disc $n+1$. After the last move of disc $n+1$, the other discs have to be brought to $\bar{t}$. This leads to a recursive construction of all shortest paths from σ to t. Lemma 6, finally, makes this construction more efficient by pointing out the advantages of leaving the intermediate states as irregular as possible.

While Example 1 revealed that even in the case of a regular initial state uniqueness of the shortest path does not hold and that there are shortest paths with two moves of the largest disc, the following example indicates that things are even more complicated now.

Example 3. $\sigma = [(2, 1), (2, 2), (0, 1), (2, 3), (0, 2)]$, $t = [1, 1, 1, 1, 2]$. Then a careful analysis shows that a path from σ to t needs at least 11 moves if disc 5 moves only once and 22 if it moves exactly twice, but there is a shortest path of length 9 where disc 5 moves three times! As in the construction of Theorem 6 (Proposition 9) disc 5 would not move but once, this example also verifies the assertions in Remark 12.

This shows that in general the number of candidates for a shortest path may still be considerable. That is not so if t is perfect. So the rest of this chapter is devoted to the final analysis of Lucas's second problem.

2.3. Uniqueness of the Solution to Lucas's Second Problem

The goal of this section is the following satisfying result.

THEOREM 7. *Let* $\rho \in \mathfrak{T}_n$ *and* $j \in \{0, 1, 2\}$. *Then the shortest path from* ρ *to* $\hat{j}$ *is unique, except for the case* $r = \hat{j} \wedge \rho \neq \hat{j}$, *when there are exactly two shortest paths, generated from each other by interchanging the roles of the elements of* $\{0, 1, 2\} \backslash \{j\}$.

As in the case of regular states, it will be important to know how often the largest disc will be moved in a shortest path.

LEMMA 7. *Let* $n \in \mathbf{N}, j \in \{0, 1, 2\}, \pi \in \Pi_{n+1}(\rho, \hat{j})$ *be shortest. Then disc* $n + 1$ *moves*

o) *not at all if* $r(n+1) = j$ *and* $h(n+1) = 1$,

i) *exactly once if* $r(n+1) \neq j$,

ii) *exactly twice if* $r(n+1) = j$ *and* $h(n+1) > 1$.

Proof. o) If there are moves of disc $n + 1$ in π, delete them all to arrive at a strictly shorter $\tilde{\pi} \in \Pi_{n+1}(\rho, \hat{j})$.

i) The possibilities of two or three moves of the largest disc $n + 1$ in a shortest path π will be excluded by constructing a strictly shorter path $\tilde{\pi}$ with only one move of disc $n + 1$.

Suppose disc $n + 1$ moves three times. Then, by Lemma 4, its sequence of moves is necessarily from $r(n+1)$ through $j \circ r(n+1)$ and again $r(n+1)$ to j. Also, if μ is the number of the last move of disc $n + 1$, π_μ is regular with $p_\mu(n+1) = j$ and $\bar{p}_\mu = \widehat{j \circ r(n+1)}^n$ and thus, by Theorem 2, $\mu_\pi = \mu + 2^n - 1$. Now carrying out the first $\mu - 1$ moves of π, skip every move of discs in U_ρ^{n+1}, then move disc $n + 1$ to j. This gives a path from ρ to $\tilde{\pi}_{\tilde{\mu}}$ with $\tilde{p}_{\tilde{\mu}}(n+1) = j$, $\tilde{h}_{\tilde{\mu}}(n+1) = 1$ and $\bar{\tilde{p}}_{\tilde{\mu}}(n) \neq j$, so that, by Proposition 9, $\hat{j}^{n+1}$ is reached in at most another $2^n - 1$ moves, resulting, as $\tilde{\mu} < \mu$ by at least two moves of disc $n + 1$, in a path from ρ to $\hat{j}$ shorter than π.

Suppose disc $n + 1$ moves twice. Then these moves, with numbers μ' and μ'' say, are necessarily from $r(n+1)$ through $j \circ r(n+1)$ to j. Carrying out only those of the first μ' moves of π with discs in $\complement\, U_\rho^{n+1}$, one arrives at a $\tilde{\pi}_{\tilde{\mu}'}$ with $\bar{\tilde{p}}_{\tilde{\mu}'} = \hat{j}^{n+1-h(n+1)}$. Leaving disc $n + 1$ at $r(n+1)$, one proceeds by carrying through those moves μ of π with $\mu' < \mu < \mu''$ and $d_\mu(\pi) \in \complement\, U_\rho^{n+1}$, but changing the roles of $r(n+1)$ and $j \circ r(n+1)$ for $i_\mu(\pi)$ and $j_\mu(\pi)$. One arrives at $\tilde{\pi}_{\tilde{\mu}''-1}$ with $\bar{\tilde{p}}_{\tilde{\mu}''-1} = \widehat{j \circ r(n+1)}^{n+1-h(n+1)}$ and $\underline{\tilde{p}}_{\tilde{\mu}''-1} = \underline{\rho}$, allowing disc $n + 1$ to be moved to j. Now, by Lemma 5 applied to $\sigma = \bar{\pi}_{\mu'}$ and $\tau = \bar{\pi}_{\mu''}$, $\bar{\pi}_{\mu''}$ is either regular on $r(n+1)$, in which case, by Proposition 9, $\mu(\tilde{\pi}_{\tilde{\mu}''}, \hat{j}) \leqslant 2^n - 1 = \mu(\pi_{\mu''}, \hat{j})$, or

$$\exists d \in \{1, \ldots, n\}, h_\sigma(d) = n \vee d > \max \complement\, U_\sigma^d : U_\tau^d = U_\sigma^d \wedge \underline{\tau_d} = \underline{\sigma_d} \wedge$$
$$\bar{\tau}^d = \widehat{r(n+1)}^{n - h_\sigma(d)};$$

but then discs in U_ρ^d have not been moved neither in the first μ'' moves of π nor in the first $\tilde{\mu}''$ moves of $\tilde{\pi}$. Let μ''' be the first move of d in π, so that $\mu''' = \mu'' + 2^{n - h_\rho(d)}$; on the other hand, state $\pi_{\mu'''}$ can be reached from $\tilde{\pi}_{\tilde{\mu}''}$ in at most $2^{n - h_\rho(d)}$ moves by Proposition 9, since for

$$\tilde{d} := \max \complement U^d_{\bar{\tilde{\pi}}_{\mu''}} : \bar{\tilde{p}}_{\tilde{\mu}''}(\tilde{d}) = r(n+1) \vee \bar{\tilde{h}}_{\tilde{\mu}''}(\tilde{d}) = 1 .$$

ii) Disc $n+1$ has to be moved at least once. After its first move, situation i is reached. □

The last step shows that the only possible ambiguity in the sequence of moves of the largest disc might arise in case ii of Lemma 7 by the question to which of the pegs $\neq j$ it should be moved. Lemma 8 answers this question.

LEMMA 8. *Let* $(i, j) \in \{0, 1, 2\}^2, i \neq j, \rho \in \mathfrak{T}_n$ *with* $r(n) = i \circ j$. *Then*

$$\mu(\rho, \hat{i}) = \mu(\rho, \hat{j}) \Leftrightarrow r = \widehat{i \circ j} .$$

Proof. "$\Leftarrow$" is trivial by interchanging i and j.

"$\Rightarrow$" will be proved by induction on n.

a) Cases $n = 0$ and $n = 1$ are trivial.

b) Suppose $\{c \in \{1, ..., n\}; r(c) \neq i \circ j\} \neq \emptyset$. Let π_i, π_j be shortest paths from ρ to $\hat{i}^{n+1}$ and $\hat{j}^{n+1}$, respectively, and let $d := \max \complement U^{n+1}_\rho$.

If $r(d) = i \circ j$, then $\mu(\rho, \hat{i}) = \mu(\bar{\rho}, \hat{j}) + 2^n$ and $\mu(\rho, \hat{j}) = \mu(\bar{\rho}, \hat{i}) + 2^n$ by Lemma 7, Lemma 3 and Theorem 2. But by induction hypothesis $\mu(\bar{\rho}, \hat{i}) \neq \mu(\bar{\rho}, \hat{j})$.

If, without loss of generality, $r(d) = i$, then in π_i leave out the first move of disc d, go on until the move of disc $n+1$ ignoring the moves of discs in U^d_ρ and interchanging i and j in the moves of the other discs; then move disc $n+1$ to j. To the rest of the moves, Lemma 6 can be applied (again interchanging i and j), yielding a path from ρ to $\hat{j}$ strictly shorter (by at least one move of disc d) than π_i. □

Now Lemmas 5 to 8 comprise all the information necessary to prove Theorem 7.

Proof of Theorem 7 by induction on n. a) Case $n = 0$ is trivial.

b) If $r(n+1) = j$ and $h(n+1) > 1$, then there are still two possible sequences of moves for disc $n+1$, differing in the intermediate peg to be passed. Let $d := \max \complement U^{n+1}_\rho$. If $r(d) = j$, then Lemma 8 can be applied. Otherwise the path which moves disc $n+1$ to $j \circ r(d)$ is strictly shorter than the one with intermediate peg $r(d)$ by an argument similar to that in the proof of Lemma 8 and with the aid of Lemma 6.

In all the other cases, the moves of disc $n+1$ are determined by Lemma 7, the moves of the other n discs are governed by Lemmas 5 and 6, and their uniqueness follows by induction hypothesis, keeping in mind that the paths of Lemma 5 are actually paths from $\bar{\sigma}^d$ to $\hat{j}^{n-h_\sigma(d)}$. □

Using the methods of this chapter, one finds the shortest path from σ to $\hat{0}$ in Figure 1 with length 102.

3. Open problems

Much of the discussion of the **TH** in computer science literature has been a controversy between recursion and iteration. It has turned out here that problems involving just regular states, can be solved by iteration very elegantly (Chapter 1). On the other hand, as soon as irregular states are considered, only recursive solutions are available (Chapter 2). While for 𝔓3 the solution is essentially unique and the recursion will work efficiently, the situation for 𝔓4 is less straightforward. Although the number of cases to be considered can be further limited by methods as in Section 2.3 (e.g. the shortest path (of length 108) from σ to r in Figure 1 is unique), and one can show that no three moves of the largest disc $n+1$ occur if $r(n+1) = t(n+1)$ and $h(n+1) > 1$, it is not clear whether there are shortest path problems with even three different solutions. Also it seems that the minimal length in 𝔓s 3 and 4 can only be determined recursively.

The only existing solution to the **TH** with more than three pegs is also recursive, and the preceeding chapters should have demonstrated that things are not as easy as many authors might hope (see the remarks in the Introduction). To move the largest disc $n+1$ in the solution of 𝔓0 with four pegs, the n other discs have to be transferred to two different pegs; after the last move of disc $n+1$, discs 1 to n have to be sent from some two pegs to the top of disc $n+1$. Again it follows by symmetry that disc $n+1$ will only be moved once in a shortest path. But this time, this does not reduce the problem for $n+1$ discs to a similar one with only n discs, but to the different setting of how to transfer n discs from a perfect state to two different pegs in the shortest possible way. Here is where the hitherto unjustified assumption made in literature enters, namely that this will be achieved by dividing the perfect state in a suitable way into two parts, then first solving 𝔓0 for the smaller discs using four pegs, leaving them untouched thereafter and solving the old problem for the larger discs using three pegs only.

The validity of this hypothesis is the most interesting open problem about the **TH**. It might be found by checking a suitable guess about the minimal length for 𝔓2 with four pegs against the recursive solution which can easily be constructed using the fact, proved as Lemma 1, that the largest disc will not move more than three times.

In contrast to this recursive solution, the use of the hypothesis leads to a very elegant iterative solution to 𝔓0 with four (or more) pegs (see Hinz [26]), resembling algorithm i in 1.2.1, with the astonishing result that the transfer of 64 discs can be carried out in less than 6 hours (compare the time needed with three pegs, indicated in the Introduction!).

To conclude, it can be said that the invention of Edouard Lucas, besides its appeal as a puzzle for human beings as well as for computer performance, has been endowed with enough structure to be treated mathematically (the problem 𝔓5 : = irregular → irregular without the "devine rule" (0) seems to have almost no mathematical structure), but not with so much to be trivial and uncapable of meaningful generalizations. As long as there are still open problems, a mathematical subject is not dead. The brahmins are alive and as long as they are still moving golden discs, the world will, according to legend, not fall to dust. Let us hope so!

REFERENCES

[1] AFRIAT, S. N. *The Ring of Linked Rings.* Duckworth (London), 1982.

[2] ALLARDICE, R. E. and A. Y. FRASER. La Tour d'Hanoï. *Proc. Edinburgh Math. Soc. 2* (1883-84), 50-53.

[3] BALL, W. W. R. *Mathematical recreations and problems of past and present times.* Macmillan (London), 1892 (2nd ed.).

[4] BENDISCH, J. Generalized Sequencing Problem "Towers of Hanoi". *Z. Oper. Res. 29* (1985), 31-45.

[5] BERMAN, G. and K. D. FRYER. *Introduction to combinatorics.* Academic Press (New York), 1972.

[6] BROUSSEAU, A. Tower of Hanoi with more pegs. *J. Recreational Math. 8* (1975-76), 169-178.

[7] CHARNIAK, E. and D. MCDERMOTT. *Introduction to Artificial Intelligence.* Addison-Wesley (Reading (Mass.)), 1986.

[8] CLAUS, N. (= E. LUCAS). *La Tour d'Hanoï, Véritable casse-tête annamite.* P. Bousrez (Tours), 1883.

[9] —— La Tour d'Hanoï, Jeu de calcul. *Science et Nature, Vol. I, N° 8* (1884), 127-128.

[10] CROWE, D. W. The *n*-dimensional cube and the Tower of Hanoi. *Amer. Math. Monthly 63* (1956), 29-30.

[11] CULL, P. and E. F. ECKLUND Jr. On the Towers of Hanoi and Generalized Towers of Hanoi Problems. *Congress. Numer. 35* (1982), 229-238.

[12] —— Towers of Hanoi and analysis of algorithms. *Amer. Math. Monthly 92* (1985), 407-420.

[13] CULL, P. and C. GERETY. Is Towers of Hanoi Really Hard? *Congress. Numer. 47* (1985), 237-242.

[14] DOMORYAD, A. P. *Mathematical Games and Pastimes.* Pergamon Press (Oxford), 1964.

[15] DUDENEY, H. E. *The Canterbury Puzzles (and other curious problems).* E. P. Dutton (New York), 1908.

[16] ER, M. C. An iterative solution to the generalized Towers of Hanoi problem. *BIT 23* (1983), 295-302.

[17] —— An analysis of the generalized Towers of Hanoi Problem. *BIT 23* (1983), 429-435.

[18] —— The Generalized Towers of Hanoi Problem. *J. Inform. Optim. Sci. 5* (1984), 89-94.

[19] FERMAT, P. DE. *Œuvres, Tome Deuxième (Correspondance).* Gauthier-Villars (Paris), 1894.

[20] FRAME, J. S. (Solution to Problem 3918). *Amer. Math. Monthly 48* (1941), 216-217.

[21] GILLISPIE, C. G. (ed.). *Dictionary of scientific biography, Volume VIII.* Scribner's Sons (New York), 1973.

[22] GRAHAM, R. L., D. E. KNUTH and O. PATASHNIK. *Concrete Mathematics.* Addison-Wesley (Reading (Mass.)), 1989.

[23] GRAVES, R. P. *Life of Sir William Rowan Hamilton, Vol. III.* Dublin University Press (Dublin), 1889.

[24] HARKIN, D. On the mathematical work of François-Edouard-Anatole Lucas. *Enseign. Math. (2) 3* (1957), 276-288.

[25] HERING, H. Zur Mathematisierung des "Turm von Hanoi"-Spiels. *MNU 26* (1973), 408-411.

[26] HINZ, A. M. An Iterative Algorithm for the Tower of Hanoi with Four Pegs. *Computing 42* (1989), 133-140.

[27] JOHNSONBAUGH, R. *Essential Discrete Mathematics.* Macmillan/Collier Macmillan (New York/London), 1987.

[28] LAVALLÉE, I. Note sur le problème des Tours de Hanoï. *Rev. Roumaine Math. Pures Appl. 30* (1985), 433-438.

[29] LONGCHAMPS, G. DE. Variétés. *J. Math. Spéciales (2) 2* (1883), 286-287.

[30] LUCAS, E. *Récréations mathématiques.* Gauthier-Villars (Paris), 1882.

[31] —— *(Discours prononcé à la distribution solennelle des prix, faite le mardi 4 août 1885).* Lycée Saint-Louis (Paris), 1885.

[32] —— *Nouveaux jeux scientifiques de M. Edouard Lucas.* La Nature 17 (1889), 2e semestre, 301-303.

[33] —— *Récréations mathématiques, tome III.* Gauthier-Villars (Paris), 1893.

[34] LUNNON, W. F. The Reve's Puzzle. *Comput. J. 29* (1986), 478.

[35] MATTHES, G. Der Einsatz des "Turm von Hanoi"-Computerprogramms zur Diagnostik von Störungen des problemlösenden Denkens bei Patienten mit erworbenen Hirnschäden. *Biomedical J. 19* (1988), 10-13.

[36] PARVILLE, H. DE. La Tour d'Hanoï et la Question du Tonkin. *La Nature 12* (1884), 285-286.

[37] PAULL, M. C. *Algorithm design, A Recursion Transformation Framework.* Wiley (New York), 1988.

[38] PETTOROSSI, A. Towers of Hanoi problems: deriving iterative solutions by program transformations. *BIT 25* (1985), 327-334.

[39] ROHL, J. S. *Recursion via Pascal.* Cambridge University Press (Cambridge), 1984.

[40] ROHL, J. S. and T. D. GEDEON. The Reve's Puzzle. *Comput. J. 29* (1986), 187-188; Corrigendum *31* (1988), 190.
[41] ROTH, T. The Tower of Brahma revisited. *J. Recreational Math.* 7 (1974), 116-119.
[42] SCARIONI, F. and H. G. SPERANZA. A probabilistic analysis of an error-correcting algorithm for the Towers of Hanoi puzzle. *Inform. Process. Lett. 18* (1984), 99-103.
[43] SCHOUTE, P. H. De Ringen van Brahma. *Eigen Haard* 1884, 274-276.
[44] SIMON, H. A. The Functional Equivalence of Problem Solving Skills. *Cognitive Psychology* 7 (1975), 268-288.
[45] STEWART, B. M. (Solution to Problem 3918). *Amer. Math. Monthly 48* (1941), 217-219.
[46] TANNERY, P. *Mémoires scientifiques, Vol. VI.* Privat/Gauthier-Villars (Toulouse/Paris), 1926.
[47] WAERDEN, B. L. van der. *Science Awakening.* Noordhoff (Groningen), 1954.
[48] WALSH, T. R. A case for iteration. *Congress. Numer. 40* (1983), 409-417.
[49] WOOD, D. The Towers of Brahma and Hanoi revisited. *J. Recreational Math. 14* (1981-82), 17-24.
[50] —— Adjudicating a Towers of Hanoi Contest. *Internat. J. Comput. Math. 14* (1983), 199-207.

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