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Artikel: ZETA FUNCTIONS AND GENUS OF QUADRATIC FORMS
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for all $n \in \mathbf{R}^*$, where we have denoted by d the quadratic form $d(x) = x^T Dx$. It is also easy to check that for all $m \in \mathbf{R}$ we have

$$\theta(m, f, \mathbf{R}) = \int_{\mathbf{R}^k} \exp\left(-\pi(y^T y) + 2\pi m i(y^T D y)\right) dy$$

$$= \left(\int_{\mathbf{R}} \exp\left(-\pi y^2(1+2im)\right) dy\right)^s \left(\int_{\mathbf{R}} \exp\left(-\pi y^2(1-2im)\right) dy\right)^{k-s}$$

$$= (1+2im)^{s/2}(1-2im)^{(k-s)/2} .$$

The following result is now clear:

THEOREM 3.4. *Let* f, g *be non-singular real quadratic forms in* k *variables. The following conditions are equivalent.*

i) $f \sim g$ *over* $\mathbf{R}$,

ii) $r(\ , f, \mathbf{R}) = r(\ , g, \mathbf{R})$,

iii) $\theta(\ , f, \mathbf{R}) = \theta(\ , g, \mathbf{R})$. □

§ 4. ADELIC REPRESENTATION MASSES

Let $\mathbf{A}$ be the ring of adeles over $\mathbf{Q}$. We identify $\mathbf{A}$ with its topological dual by defining $<n, m>$, where χ is Tate's character

$$\chi(a) = \chi_\infty(a_\infty) \cdot \prod_p \chi_p(a_p) ,$$

for any $a = (a_p) \in \mathbf{A}$. Let dn be the restricted product measure of the local measures used in the preceding sections. As is well-known, dn is also a selfdual measure. Let dx be the Haar measure on $\mathbf{A}^k$ naturally induced by dn.

A non-singular integral adelic quadratic form f in k variables with unit determinant can be identified to a collection (f_p) of non-singular integral p-adic quadratic forms in k variables such that $p \nmid \det f_p$, for almost all p.

Let Φ be the Schwartz-Bruhat function on $\mathbf{A}^k$ defined by

$$\Phi = \phi_\infty \cdot \prod_p 1_{(Z_p)^k} .$$

Let $\mathbf{A}_o := \mathbf{R} x \prod_p \mathbf{Z}_p$. We consider

$$r_\Phi(n, f, \mathbf{A}) := \lim_{U \to (n)} \left(\int_{f^{-1}(U)} \Phi(x)dx \Big/ \int_U dn \right)$$
$$= r(n_\infty, f_\infty, \mathbf{R}) \,.\, \prod_p r(n_p, f_p, \mathbf{Z}_p)\,,$$

the limit being well-defined whenever the infinite product on the right is absolutely convergent. Applying Siegel's explicit formulas for $r(n_p, f_p, \mathbf{Z}_p)$ ([13, Hilfsatz 16]), it is easy to check that the product is absolutely convergent for all $n \in \mathbf{A}_o$ if $k \geqslant 5$. Since $r(\ , f_\infty, \mathbf{R}) \in L^1(\mathbf{R})$ and $\prod_p \mathbf{Z}_p$ is compact, r_Φ is an everywhere defined continuous function on $\mathbf{A}$, with support contained in $\mathbf{A}_o$, and integrable on $\mathbf{A}$. On the other hand, clearly $\Phi \in L^1(\mathbf{A}^k)$ and we have

$$\theta_\Phi(m, f, \mathbf{A}) := \int_{\mathbf{A}^k} \Phi(x) < f(x), m > dx = \theta(m_\infty, f_\infty, \mathbf{R}) \prod \theta(m_p, f_p, \mathbf{Q}_p)\,.$$

Note that the infinite product is always well-defined since only a finite number of factors are different from 1.

We recall that given two integral adelic quadratic forms in k variables it is said that they belong to the same *genus* if $f_\infty \sim g_\infty$ over $\mathbf{R}$ and $f_p \sim g_p$ over $\mathbf{Z}_p$ for all p. We say that they are of the same 2-type if f_2 and g_2 are of the same type over $\mathbf{Z}_2$.

THEOREM 4.1. *Let f, g be two non-singular integral adelic quadratic forms in $k \geqslant 5$ variables with unit determinant. Assume that they are of the same 2-type. Then the following conditions are equivalent:*

i) $\operatorname{gen} f = \operatorname{gen} g$,

ii) $r_\Phi(\ , f, \mathbf{A}) = r_\Phi(\ , g, \mathbf{A})$,

iii) $\theta_\Phi(\ , f, \mathbf{A}) = \theta_\Phi(\ , g, \mathbf{A})$.

Proof. Two forms in the same genus have the same local integral representation masses, hence i) ⇒ ii). Since θ_Φ is just the Fourier transform of r_Φ, ii) ⇒ iii). Now condition iii) is equivalent to $\theta(\ , f_\infty, \mathbf{R}) = \theta(\ , g_\infty, \mathbf{R})$ and $\theta(\ , f_p, \mathbf{Q}_p) = \theta(\ , g_p, \mathbf{Q}_p)$ for all p; therefore by Theorems 2.3 and 3.4, iii) ⇒ i). □

We deal now with $\mathbf{A}$-equivalence of forms. If f and g are two non-singular quadratic forms defined over $\mathbf{Q}$, we have by the Minkowski-Hasse theorem that $f \sim g$ over $\mathbf{Q}$ if and only if $f \sim g$ over $\mathbf{A}$. Thus Theorem 4.2 below can be also considered as a characterization of $\mathbf{Q}$-equivalence in terms of representation masses.

For every finite set S of primes and for every integer $s \geqslant 1$ we consider the following function defined on $\mathbf{A}$:

$$r_{S,s}(n, f, \mathbf{A}) = r(n_\infty, f_\infty, \mathbf{R}) \cdot \prod_{p \in S} r_s(n_p, f_p, \mathbf{Z}_p) \cdot \prod_{p \notin S} r(n_p, f_p, \mathbf{Z}_p).$$

As before, $r_{S,s}$ is well-defined, continuous and integrable if $k \geqslant 5$. The corresponding function $\theta_{S,s}(\ , f, \mathbf{A})$ will be well-defined and continuous for all $k \geqslant 1$, being the Fourier transform of the former.

Since $f \sim g$ over $\mathbf{A}$ is equivalent to $f_p \sim g_p$ over $\mathbf{Q}_p$ for all p including $p = \infty$, and $f_p \sim g_p$ over $\mathbf{Z}_p$ for almost all p, we get from Theorem 2.3, 3.3 and 3.4 the following:

THEOREM 4.2. *Let* f, g *be two non-singular integral adelic quadratic forms in* $k \geqslant 5$ *variables. Let* $S = \{p ; p \mid \det f_p \,.\, \det g_p\}$ *and let* $s \geqslant \max(s_o(f), s_o(g))$. *Then the following conditions are equivalent:*

i) $f \sim g$ *over* $\mathbf{A}$,

ii) $r_{S,s}(\ , f, \mathbf{A}) = r_{S,s}(\ , g, \mathbf{A})$,

iii) $\theta_{S,s}(\ , f, \mathbf{A}) = \theta_{S,s}(\ , g, \mathbf{A})$. □

Note that we could have also expressed these functions as $r_{S,s} = r_{\Phi_{S,s}}$, $\theta_{S,s} = \theta_{\Phi_{S,s}}$, where $\Phi_{S,s} \in L^1(\mathbf{A}^k)$ is defined as:

$$\Phi_{S,s} = \phi_\infty \cdot \prod_{p \in S} \phi_s \cdot \prod_{p \notin S} 1_{(\mathbf{Z}_p)^k}.$$

§ 5. REPRESENTATION MASSES IN Z

Let (V, q) be a regular quadratic space over $\mathbf{Q}$ of dimension k, and let G be the proper orthogonal group of this space. The adele group $G(\mathbf{A})$ operates in the set of lattices L of V; by definition the orbit of L under this action is called the genus of L. The orbit of L under the subgroup $G(\mathbf{Q})$ of $G(\mathbf{A})$ is the class of L.

If $L = \mathbf{Z}e_1 \oplus \ldots \oplus \mathbf{Z}e_k$ is a lattice of V, the formula

$$f(x_1, \ldots, x_k) = q(x_1 e_1 + \ldots + x_k e_k)$$

stablishes a one to one correspondence between the set of classes of lattices of (V, q) and the set of classes, over $\mathbf{Z}$, of quadratic forms which are $\mathbf{Q}$-equivalent to q.