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for all $n \in \mathbf{R}^*$, where we have denoted by d the quadratic form $d(x) = x^T D x$. It is also easy to check that for all $m \in \mathbf{R}$ we have

$$\begin{aligned} \theta(m, f, \mathbf{R}) &= \int_{\mathbf{R}^k} \exp(-\pi(y^T y) + 2\pi m i(y^T D y)) dy \\ &= \left(\int_{\mathbf{R}} \exp(-\pi y^2(1 + 2im)) dy \right)^s \left(\int_{\mathbf{R}} \exp(-\pi y^2(1 - 2im)) dy \right)^{k-s} \\ &= (1 + 2im)^{s/2} (1 - 2im)^{(k-s)/2}. \end{aligned}$$

The following result is now clear:

THEOREM 3.4. *Let f, g be non-singular real quadratic forms in k variables. The following conditions are equivalent.*

- i) $f \sim g$ over \mathbf{R} ,
- ii) $r(\ , f, \mathbf{R}) = r(\ , g, \mathbf{R})$,
- iii) $\theta(\ , f, \mathbf{R}) = \theta(\ , g, \mathbf{R})$. \square

§ 4. ADELIC REPRESENTATION MASSES

Let \mathbf{A} be the ring of adeles over \mathbf{Q} . We identify \mathbf{A} with its topological dual by defining $\langle n, m \rangle$, where χ is Tate's character

$$\chi(a) = \chi_\infty(a_\infty) \cdot \prod_p \chi_p(a_p),$$

for any $a = (a_p) \in \mathbf{A}$. Let dn be the restricted product measure of the local measures used in the preceding sections. As is well-known, dn is also a selfdual measure. Let dx be the Haar measure on \mathbf{A}^k naturally induced by dn .

A non-singular integral adelic quadratic form f in k variables with unit determinant can be identified to a collection (f_p) of non-singular integral p -adic quadratic forms in k variables such that $p \nmid \det f_p$, for almost all p .

Let Φ be the Schwartz-Bruhat function on \mathbf{A}^k defined by

$$\Phi = \phi_\infty \cdot \prod_p 1_{(Z_p)^k}.$$

Let $\mathbf{A}_o := \mathbf{R}x \prod_p \mathbf{Z}_p$. We consider

$$r_{\Phi}(n, f, \mathbf{A}) := \lim_{U \rightarrow (n)} \left(\int_{f^{-1}(U)} \Phi(x) dx / \int_U dn \right) \\ = r(n_{\infty}, f_{\infty}, \mathbf{R}) \cdot \prod_p r(n_p, f_p, \mathbf{Z}_p),$$

the limit being well-defined whenever the infinite product on the right is absolutely convergent. Applying Siegel's explicit formulas for $r(n_p, f_p, \mathbf{Z}_p)$ ([13, Hilfsatz 16]), it is easy to check that the product is absolutely convergent for all $n \in \mathbf{A}_o$ if $k \geq 5$. Since $r(\cdot, f_{\infty}, \mathbf{R}) \in L^1(\mathbf{R})$ and $\prod_p \mathbf{Z}_p$ is compact, r_{Φ} is an everywhere defined continuous function on \mathbf{A} , with support contained in \mathbf{A}_o , and integrable on \mathbf{A} . On the other hand, clearly $\Phi \in L^1(\mathbf{A}^k)$ and we have

$$\theta_{\Phi}(m, f, \mathbf{A}) := \int_{\mathbf{A}^k} \Phi(x) \langle f(x), m \rangle dx = \theta(m_{\infty}, f_{\infty}, \mathbf{R}) \prod_p \theta(m_p, f_p, \mathbf{Q}_p).$$

Note that the infinite product is always well-defined since only a finite number of factors are different from 1.

We recall that given two integral adelic quadratic forms in k variables it is said that they belong to the same *genus* if $f_{\infty} \sim g_{\infty}$ over \mathbf{R} and $f_p \sim g_p$ over \mathbf{Z}_p for all p . We say that they are of the same 2-type if f_2 and g_2 are of the same type over \mathbf{Z}_2 .

THEOREM 4.1. *Let f, g be two non-singular integral adelic quadratic forms in $k \geq 5$ variables with unit determinant. Assume that they are of the same 2-type. Then the following conditions are equivalent:*

- i) $\text{gen } f = \text{gen } g$,
- ii) $r_{\Phi}(\cdot, f, \mathbf{A}) = r_{\Phi}(\cdot, g, \mathbf{A})$,
- iii) $\theta_{\Phi}(\cdot, f, \mathbf{A}) = \theta_{\Phi}(\cdot, g, \mathbf{A})$.

Proof. Two forms in the same genus have the same local integral representation masses, hence i) \Rightarrow ii). Since θ_{Φ} is just the Fourier transform of r_{Φ} , ii) \Rightarrow iii). Now condition iii) is equivalent to $\theta(\cdot, f_{\infty}, \mathbf{R}) = \theta(\cdot, g_{\infty}, \mathbf{R})$ and $\theta(\cdot, f_p, \mathbf{Q}_p) = \theta(\cdot, g_p, \mathbf{Q}_p)$ for all p ; therefore by Theorems 2.3 and 3.4, iii) \Rightarrow i). \square

We deal now with \mathbf{A} -equivalence of forms. If f and g are two non-singular quadratic forms defined over \mathbf{Q} , we have by the Minkowski-Hasse theorem that $f \sim g$ over \mathbf{Q} if and only if $f \sim g$ over \mathbf{A} . Thus Theorem 4.2 below can be also considered as a characterization of \mathbf{Q} -equivalence in terms of representation masses.

For every finite set S of primes and for every integer $s \geq 1$ we consider the following function defined on \mathbf{A} :

$$r_{S,s}(n, f, \mathbf{A}) = r(n_\infty, f_\infty, \mathbf{R}) \cdot \prod_{p \in S} r_s(n_p, f_p, \mathbf{Z}_p) \cdot \prod_{p \notin S} r(n_p, f_p, \mathbf{Z}_p).$$

As before, $r_{S,s}$ is well-defined, continuous and integrable if $k \geq 5$. The corresponding function $\theta_{S,s}(\cdot, f, \mathbf{A})$ will be well-defined and continuous for all $k \geq 1$, being the Fourier transform of the former.

Since $f \sim g$ over \mathbf{A} is equivalent to $f_p \sim g_p$ over \mathbf{Q}_p for all p including $p = \infty$, and $f_p \sim g_p$ over \mathbf{Z}_p for almost all p , we get from Theorem 2.3, 3.3 and 3.4 the following:

THEOREM 4.2. *Let f, g be two non-singular integral adelic quadratic forms in $k \geq 5$ variables. Let $S = \{p; p \mid \det f_p \cdot \det g_p\}$ and let $s \geq \max(s_o(f), s_o(g))$. Then the following conditions are equivalent:*

- i) $f \sim g$ over \mathbf{A} ,
- ii) $r_{S,s}(\cdot, f, \mathbf{A}) = r_{S,s}(\cdot, g, \mathbf{A})$,
- iii) $\theta_{S,s}(\cdot, f, \mathbf{A}) = \theta_{S,s}(\cdot, g, \mathbf{A})$. \square

Note that we could have also expressed these functions as $r_{S,s} = r_{\Phi_{S,s}}$, $\theta_{S,s} = \theta_{\Phi_{S,s}}$, where $\Phi_{S,s} \in L^1(\mathbf{A}^k)$ is defined as:

$$\Phi_{S,s} = \phi_\infty \cdot \prod_{p \in S} \phi_s \cdot \prod_{p \notin S} 1_{(\mathbf{Z}_p)^k}.$$

§ 5. REPRESENTATION MASSES IN \mathbf{Z}

Let (V, q) be a regular quadratic space over \mathbf{Q} of dimension k , and let G be the proper orthogonal group of this space. The adèle group $G(\mathbf{A})$ operates in the set of lattices L of V ; by definition the orbit of L under this action is called the genus of L . The orbit of L under the subgroup $G(\mathbf{Q})$ of $G(\mathbf{A})$ is the class of L .

If $L = \mathbf{Z}e_1 \oplus \dots \oplus \mathbf{Z}e_k$ is a lattice of V , the formula

$$f(x_1, \dots, x_k) = q(x_1e_1 + \dots + x_ke_k)$$

establishes a one to one correspondence between the set of classes of lattices of (V, q) and the set of classes, over \mathbf{Z} , of quadratic forms which are \mathbf{Q} -equivalent to q .