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for all  $n \in \mathbf{R}^*$ , where we have denoted by  $d$  the quadratic form  $d(x) = x^T D x$ . It is also easy to check that for all  $m \in \mathbf{R}$  we have

$$\begin{aligned} \theta(m, f, \mathbf{R}) &= \int_{\mathbf{R}^k} \exp(-\pi(y^T y) + 2\pi m i(y^T D y)) dy \\ &= \left( \int_{\mathbf{R}} \exp(-\pi y^2(1 + 2im)) dy \right)^s \left( \int_{\mathbf{R}} \exp(-\pi y^2(1 - 2im)) dy \right)^{k-s} \\ &= (1 + 2im)^{s/2} (1 - 2im)^{(k-s)/2}. \end{aligned}$$

The following result is now clear:

**THEOREM 3.4.** *Let  $f, g$  be non-singular real quadratic forms in  $k$  variables. The following conditions are equivalent.*

- i)  $f \sim g$  over  $\mathbf{R}$ ,
- ii)  $r(\cdot, f, \mathbf{R}) = r(\cdot, g, \mathbf{R})$ ,
- iii)  $\theta(\cdot, f, \mathbf{R}) = \theta(\cdot, g, \mathbf{R})$ .  $\square$

#### § 4. ADELIC REPRESENTATION MASSES

Let  $\mathbf{A}$  be the ring of adèles over  $\mathbf{Q}$ . We identify  $\mathbf{A}$  with its topological dual by defining  $\langle n, m \rangle$ , where  $\chi$  is Tate's character

$$\chi(a) = \chi_\infty(a_\infty) \cdot \prod_p \chi_p(a_p),$$

for any  $a = (a_p) \in \mathbf{A}$ . Let  $dn$  be the restricted product measure of the local measures used in the preceding sections. As is well-known,  $dn$  is also a selfdual measure. Let  $dx$  be the Haar measure on  $\mathbf{A}^k$  naturally induced by  $dn$ .

A non-singular integral adelic quadratic form  $f$  in  $k$  variables with unit determinant can be identified to a collection  $(f_p)$  of non-singular integral  $p$ -adic quadratic forms in  $k$  variables such that  $p \nmid \det f_p$ , for almost all  $p$ .

Let  $\Phi$  be the Schwartz-Bruhat function on  $\mathbf{A}^k$  defined by

$$\Phi = \phi_\infty \cdot \prod_p 1_{(\mathbf{Z}_p)^k}.$$

Let  $\mathbf{A}_o := \mathbf{R} \times \prod_p \mathbf{Z}_p$ . We consider

$$r_{\Phi}(n, f, \mathbf{A}) := \lim_{U \rightarrow (n)} \left( \int_{f^{-1}(U)} \Phi(x) dx / \int_U dn \right) \\ = r(n_{\infty}, f_{\infty}, \mathbf{R}) \cdot \prod_p r(n_p, f_p, \mathbf{Z}_p),$$

the limit being well-defined whenever the infinite product on the right is absolutely convergent. Applying Siegel's explicit formulas for  $r(n_p, f_p, \mathbf{Z}_p)$  ([13, Hilfsatz 16]), it is easy to check that the product is absolutely convergent for all  $n \in \mathbf{A}_o$  if  $k \geq 5$ . Since  $r(\cdot, f_{\infty}, \mathbf{R}) \in L^1(\mathbf{R})$  and  $\prod_p \mathbf{Z}_p$  is compact,  $r_{\Phi}$  is an everywhere defined continuous function on  $\mathbf{A}$ , with support contained in  $\mathbf{A}_o$ , and integrable on  $\mathbf{A}$ . On the other hand, clearly  $\Phi \in L^1(\mathbf{A}^k)$  and we have

$$\theta_{\Phi}(m, f, \mathbf{A}) := \int_{\mathbf{A}^k} \Phi(x) \langle f(x), m \rangle dx = \theta(m_{\infty}, f_{\infty}, \mathbf{R}) \prod_p \theta(m_p, f_p, \mathbf{Q}_p).$$

Note that the infinite product is always well-defined since only a finite number of factors are different from 1.

We recall that given two integral adelic quadratic forms in  $k$  variables it is said that they belong to the same *genus* if  $f_{\infty} \sim g_{\infty}$  over  $\mathbf{R}$  and  $f_p \sim g_p$  over  $\mathbf{Z}_p$  for all  $p$ . We say that they are of the same 2-type if  $f_2$  and  $g_2$  are of the same type over  $\mathbf{Z}_2$ .

**THEOREM 4.1.** *Let  $f, g$  be two non-singular integral adelic quadratic forms in  $k \geq 5$  variables with unit determinant. Assume that they are of the same 2-type. Then the following conditions are equivalent:*

- i)  $\text{gen } f = \text{gen } g$ ,
- ii)  $r_{\Phi}(\cdot, f, \mathbf{A}) = r_{\Phi}(\cdot, g, \mathbf{A})$ ,
- iii)  $\theta_{\Phi}(\cdot, f, \mathbf{A}) = \theta_{\Phi}(\cdot, g, \mathbf{A})$ .

*Proof.* Two forms in the same genus have the same local integral representation masses, hence i)  $\Rightarrow$  ii). Since  $\theta_{\Phi}$  is just the Fourier transform of  $r_{\Phi}$ , ii)  $\Rightarrow$  iii). Now condition iii) is equivalent to  $\theta(\cdot, f_{\infty}, \mathbf{R}) = \theta(\cdot, g_{\infty}, \mathbf{R})$  and  $\theta(\cdot, f_p, \mathbf{Q}_p) = \theta(\cdot, g_p, \mathbf{Q}_p)$  for all  $p$ ; therefore by Theorems 2.3 and 3.4, iii)  $\Rightarrow$  i).  $\square$

We deal now with  $\mathbf{A}$ -equivalence of forms. If  $f$  and  $g$  are two non-singular quadratic forms defined over  $\mathbf{Q}$ , we have by the Minkowski-Hasse theorem that  $f \sim g$  over  $\mathbf{Q}$  if and only if  $f \sim g$  over  $\mathbf{A}$ . Thus Theorem 4.2 below can be also considered as a characterization of  $\mathbf{Q}$ -equivalence in terms of representation masses.

For every finite set  $S$  of primes and for every integer  $s \geq 1$  we consider the following function defined on  $\mathbf{A}$ :

$$r_{S,s}(n, f, \mathbf{A}) = r(n_\infty, f_\infty, \mathbf{R}) \cdot \prod_{p \in S} r_s(n_p, f_p, \mathbf{Z}_p) \cdot \prod_{p \notin S} r(n_p, f_p, \mathbf{Z}_p).$$

As before,  $r_{S,s}$  is well-defined, continuous and integrable if  $k \geq 5$ . The corresponding function  $\theta_{S,s}(\cdot, f, \mathbf{A})$  will be well-defined and continuous for all  $k \geq 1$ , being the Fourier transform of the former.

Since  $f \sim g$  over  $\mathbf{A}$  is equivalent to  $f_p \sim g_p$  over  $\mathbf{Q}_p$  for all  $p$  including  $p = \infty$ , and  $f_p \sim g_p$  over  $\mathbf{Z}_p$  for almost all  $p$ , we get from Theorem 2.3, 3.3 and 3.4 the following:

**THEOREM 4.2.** *Let  $f, g$  be two non-singular integral adelic quadratic forms in  $k \geq 5$  variables. Let  $S = \{p; p \mid \det f_p \cdot \det g_p\}$  and let  $s \geq \max(s_o(f), s_o(g))$ . Then the following conditions are equivalent:*

- i)  $f \sim g$  over  $\mathbf{A}$ ,
- ii)  $r_{S,s}(\cdot, f, \mathbf{A}) = r_{S,s}(\cdot, g, \mathbf{A})$ ,
- iii)  $\theta_{S,s}(\cdot, f, \mathbf{A}) = \theta_{S,s}(\cdot, g, \mathbf{A})$ .  $\square$

Note that we could have also expressed these functions as  $r_{S,s} = r_{\Phi_{S,s}}$ ,  $\theta_{S,s} = \theta_{\Phi_{S,s}}$ , where  $\Phi_{S,s} \in L^1(\mathbf{A}^k)$  is defined as:

$$\Phi_{S,s} = \phi_\infty \cdot \prod_{p \in S} \phi_s \cdot \prod_{p \notin S} 1_{(\mathbf{Z}_p)^k}.$$

## § 5. REPRESENTATION MASSES IN $\mathbf{Z}$

Let  $(V, q)$  be a regular quadratic space over  $\mathbf{Q}$  of dimension  $k$ , and let  $G$  be the proper orthogonal group of this space. The adèle group  $G(\mathbf{A})$  operates in the set of lattices  $L$  of  $V$ ; by definition the orbit of  $L$  under this action is called the genus of  $L$ . The orbit of  $L$  under the subgroup  $G(\mathbf{Q})$  of  $G(\mathbf{A})$  is the class of  $L$ .

If  $L = \mathbf{Z}e_1 \oplus \dots \oplus \mathbf{Z}e_k$  is a lattice of  $V$ , the formula

$$f(x_1, \dots, x_k) = q(x_1e_1 + \dots + x_ke_k)$$

establishes a one to one correspondence between the set of classes of lattices of  $(V, q)$  and the set of classes, over  $\mathbf{Z}$ , of quadratic forms which are  $\mathbf{Q}$ -equivalent to  $q$ .