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§ 3. Local representation masses and $\mathbf{Q}_p$-equivalence of forms

There is a formula due to Minkowski (cf. [9]) for $\theta(u, f, p^t)$ if t is large enough, in which appear the well-known pair of invariants determining the $\mathbf{Q}_p$-equivalence class of f. As a consequence of this formula (cf. Proposition 3.2), we shall obtain a characterization of $\mathbf{Q}_p$-equivalence of forms through local representation masses.

Let $s \geqslant 1$ be an integer and $X_s = \{m \in \mathbf{Q}_p \mid -v_p(m) > s\}$. For any integral p-adic quadratic form f we define the functions:

$$r_s(\ ,f, \mathbf{Z}_p) = p^{-\delta(f)/2}(r(\ ,f,\mathbf{Z}_p) - p^{(1-k)s} r(\ ,f,p^s)),$$
$$\theta_s(\ ,f,\mathbf{Q}_p) = p^{-\delta(f)/2}\theta(\ ,f,\mathbf{Q}_p) \cdot 1_{X_s},$$

where $\delta(f) = v_p(\det f)$.

The reader may check that the function defined on $\mathbf{Q}_p^k \backslash \{0\}$ by

$$\phi_s(x) = p^{-\delta(f)/2}\left(1 - p^{(1-k)s}\frac{r(f(x), f, p^s)}{r(f(x), f, \mathbf{Z}_p)}\right) \cdot 1_{(\mathbf{Z}_p)^k}(x)$$

is integrable over $\mathbf{Q}_p$ and that $r_s = r_{\phi_s}$, $\theta_s = \theta_{\phi_s}$, so that these functions follow the general pattern mentioned in the introduction. Note that ϕ_s is not a Schwartz-Bruhat function.

Proposition 3.1. *$r_s \in L^1(\mathbf{Z}_p)$ and $\theta_s(m) = \int_{\mathbf{Z}_p} r_s(n) <m, n> dn$.*

Proof. r_s is integrable since r and $r\,(\text{mod } p^s)$ are integrable. To prove the second assertion, by Proposition 2.2 we need only to compute

$$\hat{r}(m, f, p^s) = \int_{\mathbf{Z}_p} r(n, f, p^s) <m, n> dn\,.$$

Let $m = p^{-t}u$, $u \in \mathbf{Z}_p$, $p \nmid u$, $t \geqslant 0$. Let $t_o = \max\{s, t\}$. On each class $a + p^{t_o}\mathbf{Z}_p$, the integrand is constant and we have

$$\hat{r}(m, f, p^s) = p^{-t_o} \sum_{a \in \mathbf{Z}/p^{t_o}\mathbf{Z}} r(a, f, p^s) \exp(2\pi i u a p^{-t})\,.$$

If $t \leqslant s$ we have directly:

$$p^{(1-k)s}\,\hat{r}(m, f, p^s) = \theta(up^{s-t}, f, p^s) = \theta(m, f, \mathbf{Q}_p)\,.$$

If $t > s$ the sum is equal to

$$p^{-t} \sum_{a_o \in \mathbf{Z}/p^s\mathbf{Z}} r(a_o, f, p^s) \exp(2\pi i u a_o p^{-t}) \sum_{b \in \mathbf{Z}/p^{t-s}\mathbf{Z}} \exp(2\pi i u p^{s-t})^b = 0 . \qquad \square$$

In order to simplify Minkowski's formula for the theta-values, we will make use of the invariant $[f]_p$ of a p-adic quadratic form introduced by Conway [2]. Let α_k be the last invariant factor of f and let $s_o(f) = v_p(2p\alpha_k)$.

PROPOSITION 3.2. *Let f be a non singular p-adic integral quadratic form in k variables. For all $t \geqslant s_o(f)$ and $u \in \mathbf{Z}_p^*$ we have:*

$$\theta(u, f, p^t) = p^{(\delta+kt)/2} \varepsilon_p^{t^2(k+2\delta)} \left(\frac{u}{p}\right)^{kt+\delta} [f]_p \left(\frac{d_o}{p}\right)^t , \quad \text{if} \quad p \neq 2 ,$$

$$\theta(u, f, 2^t) = 2^{(\delta+k(t+1))/2} \exp(2\pi i k/8) [f]_2 \left(\frac{2}{d_o}\right)^t \left(\frac{2}{u}\right)^{kt} [u]_2^k (u, \det f)_2 ,$$

$$\text{if} \quad p = 2 .$$

Here $\delta = \delta(f)$, $d_o = p^{-\delta} \det f$ and $(a, b)_p$ denotes Hilbert's symbol.

Proof. Since $\theta(u, f, p^t) = \theta(1, uf, p^t)$, it is easy to reduce the claims to the case $u = 1$. Assume first $p > 2$. Let $v = v_1 \dots v_r$, $w = w_1 \dots w_{k-r}$, where

$$f \sim \perp_{1 \leqslant i \leqslant v} \langle p^{s_i} v_i \rangle \perp_{1 \leqslant j \leqslant k-r} \langle p^{t_j} w_j \rangle$$

over $\mathbf{Z}_p$, with s_i odd, t_j even, $v_i, w_j \in \mathbf{Z}_p^*$ for all i, j. Let $t > \max_{i,j} \{s_i, t_j\}$; by Prop. 1.1 we have

$$\theta(1, f, p^t) = p^{(\delta+kt)/2} \begin{cases} \varepsilon_p^r \left(\dfrac{v}{p}\right) & \text{if} \quad t \text{ even} \\[2ex] \varepsilon_p^{k-r} \left(\dfrac{w}{p}\right) & \text{if} \quad t \text{ odd} . \end{cases}$$

Since $[f]_p = \varepsilon_p^r \left(\frac{v}{p}\right)$, we get the desired formula.

We deal now with the case $p = 2$. Assume that, over $\mathbf{Z}_2$,

$$f \sim \perp_{1 \leqslant i \leqslant r} \langle 2^{s_i} H_i \rangle \perp_{1 \leqslant j \leqslant k-2r} \langle 2^{t_j} v_j \rangle ,$$

where H_i is 2-dimensional improperly primitive and $v_j \in \mathbf{Z}_2^*$. Let

$$U = \perp_{s_i \text{ even}} \langle H_i \rangle , \; U' = \perp_{s_i \text{ odd}} \langle H_i \rangle , \; V = \perp_{t_j \text{ even}} \langle v_j \rangle , \; V' = \perp_{t_j \text{ odd}} \langle v_j \rangle .$$

Let d, d', v, v' denote the respective determinants of U, U', V and V'. By Proposition 1.1 we have for all $t > 1 + \max_{i,j} \{s_i, t_j\}$

$$\theta(1, f, 2^t) = 2^{(\delta + k(t+1))/2} \exp(2\pi i w/8) \left(\frac{2}{dv}\right) \left(\frac{2}{d_o}\right)^t ,$$

where $w = \sum_{1 \leqslant j \leqslant k-2r} v_j$. Let s denote the dimension of U; one can see that

$$[U]_2 = \left(\frac{2}{d}\right)(-i)^{s/2}, \qquad [2U']_2 = (-i)^{(2r-s)/2} .$$

Let m be the number of v_j's in V congruent to 3 (mod 4), and let n_1, n_3, n_5, n_7 be the respective number of v_j's in V' congruent to 1, 3, 5 or 7 (mod 8); we have

$$[V]_2 \, [2V']_2 = i^{3n_1 + n_3 + 2n_5 + 3n_7} .$$

Summing up these expressions the result follows. □

Whereas $\mathbf{Z}_p$-equivalence of forms is determined by all functions $r(\ , f, p^t)$, $t \geqslant 1$ (Theorem 1.2), or equivalently by its limit value $r(\ , f, \mathbf{Z}_p)$ (Theorem 2.3), we prove in the next theorem that $\mathbf{Q}_p$-equivalent forms are characterized by having the same differences $r_s(\ , f, \mathbf{Z}_p)$ between these two functions, for s sufficiently large.

THEOREM 3.3. *Let f, g be non singular integral p-adic quadratic forms in k variables. Suppose that* $s \geqslant \max(s_o(f), s_o(g))$. *Then the following conditions are equivalent:*

i) $f \sim g$ *over* $\mathbf{Q}_p$,

ii) $r_s(\ , f, \mathbf{Z}_p) = r_s(\ , g, \mathbf{Z}_p)$,

iii) $\theta_s(\ , f, \mathbf{Q}_p) = \theta_s(\ , g, \mathbf{Q}_p)$.

Proof. For any integer $t \geqslant 1$ we consider the difference

$$\Delta r(n, f, p^t) := p^{(1-k)(t+1)} r(n, f, p^{t+1}) - p^{(1-k)t} r(n, f, p^t) .$$

It is clear from the definitions that

$$r_s(n, f, \mathbf{Z}_p) = p^{-\delta(f)/2} \sum_{t \geqslant s} \Delta r(n, f, p^t) .$$

If f and g are $\mathbf{Q}_p$-equivalent, then Proposition 3.2 implies that

$$p^{-\delta(f)/2} \theta(u, f, p^t) = p^{-\delta(g)/2} \theta(u, g, p^t) ,$$

for all $u \in \mathbf{Z}_p^*$, $t \geqslant s$. Let $n \in \mathbf{Z}_p$, since

$$\sum_{u \in (\mathbf{Z}/p^t\mathbf{Z})^*} p^{-t}\theta(u, f, p^t) \exp(-2\pi i n u p^{-t}) = r(n, f, p^t) - p^{k-1} r(n, f, p^{t-1})$$
$$= p^{(k-1)t} \Delta r(n, f, p^{t-1}),$$

we see at once that i) $\Rightarrow$ ii). By Proposition 3.1, ii) $\Rightarrow$ iii).

Assume now condition iii). Let $t = s, s+1$ and let $u \in \mathbf{Z}_p^*$; from the equality $\theta_s(up^{-t}, f, \mathbf{Q}_p) = \theta_s(up^{-t}, g, \mathbf{Q}_p)$ it follows, using Proposition 3.2, that $[f]_p = [g]_p$ and $\left(\frac{d_o(f)}{p}\right) = \left(\frac{d_o(g)}{p}\right)$. Since the forms f and g have the same discriminant and Conway invariant, they are equivalent over $\mathbf{Q}_p$. □

Next we devote a few lines to $\mathbf{R}$-equivalence. We identify $\mathbf{R}$ with its topological dual by defining $<n, m> = \chi_\infty(n, m) := \exp(-2\pi i n m)$, for all $n, m \in \mathbf{R}$. We denote by dn, dx the Lebesgue measure on $\mathbf{R}$ and $\mathbf{R}^k$, respectively.

Let f be a non-singular real quadratic form in k variables with signature $(l, k-l)$. Let A be the matrix of f and let C be any matrix satisfying:

$$C^T A C = D, \quad D = \left(\begin{array}{c|c} I_l & 0 \\ \hline 0 & -I_{k-l} \end{array}\right).$$

$P := (CC^T)^{-1}$ is called a *majorant* of f. Since P is positive definite, the function

$$\phi_\infty(x) = |\det f|^{1/2} \exp(-\pi(x^T P x))$$

is a Schwartz function on $\mathbf{R}^k$. On $\mathbf{R}^*$ we define the functions

$$r(n, f, \mathbf{R}) = \lim_{U \to \{n\}} \left(\int_{f^{-1}(U)} \phi_\infty(x) dx / \mathrm{vol}\, U \right),$$
$$\theta(m, f, \mathbf{R}) = \int_{\mathbf{R}^k} \phi_\infty(x) <f(x), m> dx .$$

We have seen at the end of Section 2 that $r(\ , f, \mathbf{R})$ is a continuous function on $\mathbf{R}^*$, integrable on $\mathbf{R}$ and that $\theta(\ , f, \mathbf{R})$ is its Fourier transform. These functions do not depend on the chosen matrix C; they depend only on the signature of f. In fact, since $|\det C| = |\det f|^{-1/2}$, if we make the change of variables $x = Cy$ we obtain:

$$r(n, f, \mathbf{R}) = \lim_{U \to \{n\}} \left(\int_{d^{-1}(U)} \exp(-\pi(y^T y)) dy / dn(U) \right),$$

for all $n \in \mathbf{R}^*$, where we have denoted by d the quadratic form $d(x) = x^T D x$. It is also easy to check that for all $m \in \mathbf{R}$ we have

$$\theta(m, f, \mathbf{R}) = \int_{\mathbf{R}^k} \exp\left(-\pi(y^T y) + 2\pi m i (y^T D y)\right) dy$$

$$= \left(\int_{\mathbf{R}} \exp\left(-\pi y^2 (1+2im)\right) dy\right)^s \left(\int_{\mathbf{R}} \exp\left(-\pi y^2(1-2im)\right) dy\right)^{k-s}$$

$$= (1+2im)^{s/2}(1-2im)^{(k-s)/2}.$$

The following result is now clear:

THEOREM 3.4. *Let* f, g *be non-singular real quadratic forms in* k *variables. The following conditions are equivalent.*

i) $f \sim g$ *over* $\mathbf{R}$,

ii) $r(\ , f, \mathbf{R}) = r(\ , g, \mathbf{R})$,

iii) $\theta(\ , f, \mathbf{R}) = \theta(\ , g, \mathbf{R})$. □

§ 4. ADELIC REPRESENTATION MASSES

Let $\mathbf{A}$ be the ring of adeles over $\mathbf{Q}$. We identify $\mathbf{A}$ with its topological dual by defining $<n, m>$, where χ is Tate's character

$$\chi(a) = \chi_\infty(a_\infty) \cdot \prod_p \chi_p(a_p),$$

for any $a = (a_p) \in \mathbf{A}$. Let dn be the restricted product measure of the local measures used in the preceding sections. As is well-known, dn is also a selfdual measure. Let dx be the Haar measure on $\mathbf{A}^k$ naturally induced by dn.

A non-singular integral adelic quadratic form f in k variables with unit determinant can be identified to a collection (f_p) of non-singular integral p-adic quadratic forms in k variables such that $p \nmid \det f_p$, for almost all p.

Let Φ be the Schwartz-Bruhat function on $\mathbf{A}^k$ defined by

$$\Phi = \phi_\infty \cdot \prod_p 1_{(Z_p)^k}.$$

Let $\mathbf{A}_o := \mathbf{R} x \prod_p \mathbf{Z}_p$. We consider