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<b>Autor:</b>	Bayer, Pilar / Nart, Enric
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§ 3. LOCAL REPRESENTATION MASSES AND  $\mathbf{Q}_p$ -EQUIVALENCE  
OF FORMS

There is a formula due to Minkowski (cf. [9]) for  $\theta(u, f, p^t)$  if  $t$  is large enough, in which appear the well-known pair of invariants determining the  $\mathbf{Q}_p$ -equivalence class of  $f$ . As a consequence of this formula (cf. Proposition 3.2), we shall obtain a characterization of  $\mathbf{Q}_p$ -equivalence of forms through local representation masses.

Let  $s \geq 1$  be an integer and  $X_s = \{m \in \mathbf{Q}_p \mid -v_p(m) > s\}$ . For any integral  $p$ -adic quadratic form  $f$  we define the functions:

$$\begin{aligned} r_s(\ , f, \mathbf{Z}_p) &= p^{-\delta(f)/2}(r(\ , f, \mathbf{Z}_p) - p^{(1-k)s}r(\ , f, p^s)), \\ \theta_s(\ , f, \mathbf{Q}_p) &= p^{-\delta(f)/2}\theta(\ , f, \mathbf{Q}_p) \cdot 1_{X_s}, \end{aligned}$$

where  $\delta(f) = v_p(\det f)$ .

The reader may check that the function defined on  $\mathbf{Q}_p^k \setminus \{0\}$  by

$$\phi_s(x) = p^{-\delta(f)/2} \left( 1 - p^{(1-k)s} \frac{r(f(x), f, p^s)}{r(f(x), f, \mathbf{Z}_p)} \right) \cdot 1_{(\mathbf{Z}_p)^k}(x)$$

is integrable over  $\mathbf{Q}_p$  and that  $r_s = r_{\phi_s}$ ,  $\theta_s = \theta_{\phi_s}$ , so that these functions follow the general pattern mentioned in the introduction. Note that  $\phi_s$  is not a Schwartz-Bruhat function.

**PROPOSITION 3.1.**  $r_s \in L^1(\mathbf{Z}_p)$  and  $\theta_s(m) = \int_{\mathbf{Z}_p} r_s(n) \langle m, n \rangle dn$ .

*Proof.*  $r_s$  is integrable since  $r$  and  $r \pmod{p^s}$  are integrable. To prove the second assertion, by Proposition 2.2 we need only to compute

$$\hat{r}(m, f, p^s) = \int_{\mathbf{Z}_p} r(n, f, p^s) \langle m, n \rangle dn.$$

Let  $m = p^{-t}u$ ,  $u \in \mathbf{Z}_p$ ,  $p \nmid u$ ,  $t \geq 0$ . Let  $t_o = \max\{s, t\}$ . On each class  $a + p^{t_o}\mathbf{Z}_p$ , the integrand is constant and we have

$$\hat{r}(m, f, p^s) = p^{-t_o} \sum_{a \in \mathbf{Z}/p^{t_o}\mathbf{Z}} r(a, f, p^s) \exp(2\pi i u a p^{-t}).$$

If  $t \leq s$  we have directly:

$$p^{(1-k)s} \hat{r}(m, f, p^s) = \theta(up^{s-t}, f, p^s) = \theta(m, f, \mathbf{Q}_p).$$

If  $t > s$  the sum is equal to

$$p^{-t} \sum_{a_o \in \mathbf{Z}/p^s \mathbf{Z}} r(a_o, f, p^s) \exp(2\pi i u a_o p^{-t}) \sum_{b \in \mathbf{Z}/p^{t-s} \mathbf{Z}} \exp(2\pi i u p^{s-t})^b = 0. \quad \square$$

In order to simplify Minkowski's formula for the theta-values, we will make use of the invariant  $[f]_p$  of a  $p$ -adic quadratic form introduced by Conway [2]. Let  $\alpha_k$  be the last invariant factor of  $f$  and let  $s_o(f) = v_p(2p\alpha_k)$ .

**PROPOSITION 3.2.** *Let  $f$  be a non singular  $p$ -adic integral quadratic form in  $k$  variables. For all  $t \geq s_o(f)$  and  $u \in \mathbf{Z}_p^*$  we have:*

$$\begin{aligned} \theta(u, f, p^t) &= p^{(\delta+kt)/2} \varepsilon_p^{t^2(k+2\delta)} \left(\frac{u}{p}\right)^{kt+\delta} [f]_p \left(\frac{d_o}{p}\right)^t, \quad \text{if } p \neq 2, \\ \theta(u, f, 2^t) &= 2^{(\delta+k(t+1))/2} \exp(2\pi ik/8) [f]_2 \left(\frac{2}{d_o}\right)^t \left(\frac{2}{u}\right)^{kt} [u]_2^k (u, \det f)_2, \\ &\quad \text{if } p = 2. \end{aligned}$$

Here  $\delta = \delta(f)$ ,  $d_o = p^{-\delta} \det f$  and  $(a, b)_p$  denotes Hilbert's symbol.

*Proof.* Since  $\theta(u, f, p^t) = \theta(1, uf, p^t)$ , it is easy to reduce the claims to the case  $u = 1$ . Assume first  $p > 2$ . Let  $v = v_1 \dots v_r, w = w_1 \dots w_{k-r}$ , where

$$f \sim \bigwedge_{1 \leq i \leq v} \langle p^{s_i} v_i \rangle \bigwedge_{1 \leq j \leq k-r} \langle p^{t_j} w_j \rangle$$

over  $\mathbf{Z}_p$ , with  $s_i$  odd,  $t_j$  even,  $v_i, w_j \in \mathbf{Z}_p^*$  for all  $i, j$ . Let  $t > \max_{i,j} \{s_i, t_j\}$ ; by Prop. 1.1 we have

$$\theta(1, f, p^t) = p^{(\delta+kt)/2} \begin{cases} \varepsilon_p^r \left(\frac{v}{p}\right) & \text{if } t \text{ even} \\ \varepsilon_p^{k-r} \left(\frac{w}{p}\right) & \text{if } t \text{ odd.} \end{cases}$$

Since  $[f]_p = \varepsilon_p^r \left(\frac{v}{p}\right)$ , we get the desired formula.

We deal now with the case  $p = 2$ . Assume that, over  $\mathbf{Z}_2$ ,

$$f \sim \bigwedge_{1 \leq i \leq r} \langle 2^{s_i} H_i \rangle \bigwedge_{1 \leq j \leq k-2r} \langle 2^{t_j} v_j \rangle,$$

where  $H_i$  is 2-dimensional improperly primitive and  $v_j \in \mathbf{Z}_2^*$ . Let

$$U = \bigwedge_{s_i \text{ even}} \langle H_i \rangle, U' = \bigwedge_{s_i \text{ odd}} \langle H_i \rangle, V = \bigwedge_{t_j \text{ even}} \langle v_j \rangle, V' = \bigwedge_{t_j \text{ odd}} \langle v_j \rangle.$$

Let  $d, d', v, v'$  denote the respective determinants of  $U, U', V$  and  $V'$ . By Proposition 1.1 we have for all  $t > 1 + \max_{i,j} \{s_i, t_j\}$

$$\theta(1, f, 2^t) = 2^{(\delta+k(t+1))/2} \exp(2\pi i w/8) \left(\frac{2}{dv}\right) \left(\frac{2}{d_o}\right)^t,$$

where  $w = \sum_{1 \leq j \leq k-2r} v_j$ . Let  $s$  denote the dimension of  $U$ ; one can see that

$$[U]_2 = \left(\frac{2}{d}\right) (-i)^{s/2}, \quad [2U']_2 = (-i)^{(2r-s)/2}.$$

Let  $m$  be the number of  $v_j$ 's in  $V$  congruent to 3 (mod 4), and let  $n_1, n_3, n_5, n_7$  be the respective number of  $v_j$ 's in  $V'$  congruent to 1, 3, 5 or 7 (mod 8); we have

$$[V]_2 [2V']_2 = i^{3n_1+n_3+2n_5+3n_7}.$$

Summing up these expressions the result follows.  $\square$

Whereas  $\mathbf{Z}_p$ -equivalence of forms is determined by all functions  $r(\ , f, p^t)$ ,  $t \geq 1$  (Theorem 1.2), or equivalently by its limit value  $r(\ , f, \mathbf{Z}_p)$  (Theorem 2.3), we prove in the next theorem that  $\mathbf{Q}_p$ -equivalent forms are characterized by having the same differences  $r_s(\ , f, \mathbf{Z}_p)$  between these two functions, for  $s$  sufficiently large.

**THEOREM 3.3.** *Let  $f, g$  be non singular integral  $p$ -adic quadratic forms in  $k$  variables. Suppose that  $s \geq \max(s_o(f), s_o(g))$ . Then the following conditions are equivalent:*

- i)  $f \sim g$  over  $\mathbf{Q}_p$ ,
- ii)  $r_s(\ , f, \mathbf{Z}_p) = r_s(\ , g, \mathbf{Z}_p)$ ,
- iii)  $\theta_s(\ , f, \mathbf{Q}_p) = \theta_s(\ , g, \mathbf{Q}_p)$ .

*Proof.* For any integer  $t \geq 1$  we consider the difference

$$\Delta r(n, f, p^t) := p^{(1-k)(t+1)} r(n, f, p^{t+1}) - p^{(1-k)t} r(n, f, p^t).$$

It is clear from the definitions that

$$r_s(n, f, \mathbf{Z}_p) = p^{-\delta(f)/2} \sum_{t \geq s} \Delta r(n, f, p^t).$$

If  $f$  and  $g$  are  $\mathbf{Q}_p$ -equivalent, then Proposition 3.2 implies that

$$p^{-\delta(f)/2} \theta(u, f, p^t) = p^{-\delta(g)/2} \theta(u, g, p^t),$$

for all  $u \in \mathbf{Z}_p^*$ ,  $t \geq s$ . Let  $n \in \mathbf{Z}_p$ , since

$$\begin{aligned} \sum_{u \in (\mathbf{Z}/p^t\mathbf{Z})^*} p^{-t}\theta(u, f, p^t) \exp(-2\pi i n u p^{-t}) &= r(n, f, p^t) - p^{k-1}r(n, f, p^{t-1}) \\ &= p^{(k-1)t} \Delta r(n, f, p^{t-1}), \end{aligned}$$

we see at once that i)  $\Rightarrow$  ii). By Proposition 3.1, ii)  $\Rightarrow$  iii).

Assume now condition iii). Let  $t = s, s+1$  and let  $u \in \mathbf{Z}_p^*$ ; from the equality  $\theta_s(up^{-t}, f, \mathbf{Q}_p) = \theta_s(up^{-t}, g, \mathbf{Q}_p)$  it follows, using Proposition 3.2, that  $[f]_p = [g]_p$  and  $\left(\frac{d_o(f)}{p}\right) = \left(\frac{d_o(g)}{p}\right)$ . Since the forms  $f$  and  $g$  have the same discriminant and Conway invariant, they are equivalent over  $\mathbf{Q}_p$ .  $\square$

Next we devote a few lines to  $\mathbf{R}$ -equivalence. We identify  $\mathbf{R}$  with its topological dual by defining  $\langle n, m \rangle = \chi_\infty(n, m) := \exp(-2\pi i nm)$ , for all  $n, m \in \mathbf{R}$ . We denote by  $dn, dx$  the Lebesgue measure on  $\mathbf{R}$  and  $\mathbf{R}^k$ , respectively.

Let  $f$  be a non-singular real quadratic form in  $k$  variables with signature  $(l, k-l)$ . Let  $A$  be the matrix of  $f$  and let  $C$  be any matrix satisfying:

$$C^T AC = D, \quad D = \begin{pmatrix} I_l & 0 \\ 0 & -I_{k-l} \end{pmatrix}.$$

$P := (CC^T)^{-1}$  is called a *majorant* of  $f$ . Since  $P$  is positive definite, the function

$$\phi_\infty(x) = |\det f|^{1/2} \exp(-\pi(x^T P x))$$

is a Schwartz function on  $\mathbf{R}^k$ . On  $\mathbf{R}^*$  we define the functions

$$\begin{aligned} r(n, f, \mathbf{R}) &= \lim_{U \rightarrow \{n\}} \left( \int_{f^{-1}(U)} \phi_\infty(x) dx / \text{vol } U \right), \\ \theta(m, f, \mathbf{R}) &= \int_{\mathbf{R}^k} \phi_\infty(x) \langle f(x), m \rangle dx. \end{aligned}$$

We have seen at the end of Section 2 that  $r(\ , f, \mathbf{R})$  is a continuous function on  $\mathbf{R}^*$ , integrable on  $\mathbf{R}$  and that  $\theta(\ , f, \mathbf{R})$  is its Fourier transform. These functions do not depend on the chosen matrix  $C$ ; they depend only on the signature of  $f$ . In fact, since  $|\det C| = |\det f|^{-1/2}$ , if we make the change of variables  $x = Cy$  we obtain:

$$r(n, f, \mathbf{R}) = \lim_{U \rightarrow \{n\}} \left( \int_{d^{-1}(U)} \exp(-\pi(y^T y)) dy / dn(U) \right),$$

for all  $n \in \mathbf{R}^*$ , where we have denoted by  $d$  the quadratic form  $d(x) = x^T D x$ . It is also easy to check that for all  $m \in \mathbf{R}$  we have

$$\begin{aligned}\theta(m, f, \mathbf{R}) &= \int_{\mathbf{R}^k} \exp(-\pi(y^T y) + 2\pi m i(y^T D y)) dy \\ &= \left( \int_{\mathbf{R}} \exp(-\pi y^2(1+2im)) dy \right)^s \left( \int_{\mathbf{R}} \exp(-\pi y^2(1-2im)) dy \right)^{k-s} \\ &= (1+2im)^{s/2}(1-2im)^{(k-s)/2}.\end{aligned}$$

The following result is now clear:

**THEOREM 3.4.** *Let  $f, g$  be non-singular real quadratic forms in  $k$  variables. The following conditions are equivalent.*

- i)  $f \sim g$  over  $\mathbf{R}$ ,
- ii)  $r(\ , f, \mathbf{R}) = r(\ , g, \mathbf{R})$ ,
- iii)  $\theta(\ , f, \mathbf{R}) = \theta(\ , g, \mathbf{R})$ .  $\square$

#### § 4. ADELIC REPRESENTATION MASSES

Let  $\mathbf{A}$  be the ring of adeles over  $\mathbf{Q}$ . We identify  $\mathbf{A}$  with its topological dual by defining  $\langle n, m \rangle$ , where  $\chi$  is Tate's character

$$\chi(a) = \chi_\infty(a_\infty) \cdot \prod_p \chi_p(a_p),$$

for any  $a = (a_p) \in \mathbf{A}$ . Let  $dn$  be the restricted product measure of the local measures used in the preceding sections. As is well-known,  $dn$  is also a selfdual measure. Let  $dx$  be the Haar measure on  $\mathbf{A}^k$  naturally induced by  $dn$ .

A non-singular integral adelic quadratic form  $f$  in  $k$  variables with unit determinant can be identified to a collection  $(f_p)$  of non-singular integral  $p$ -adic quadratic forms in  $k$  variables such that  $p \nmid \det f_p$ , for almost all  $p$ .

Let  $\Phi$  be the Schwartz-Bruhat function on  $\mathbf{A}^k$  defined by

$$\Phi = \phi_\infty \cdot \prod_p 1_{(\mathbf{Z}_p)^k}.$$

Let  $\mathbf{A}_o := \mathbf{R}x \prod_p \mathbf{Z}_p$ . We consider