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possible anymore, as the following example shows: By Proposition 1.1 the diagonal forms $f = <1, 2, 2, 2, 4>$, $g = <1, 1, 2, 4, 4>$ have the same Gauss sums $\theta(\ , f, 2^t) = \theta(\ , g, 2^t)$ for all $t \geqslant 1$, however they are obviously not $\mathbf{Z}_2$-equivalent.

The theory of Minkowski reproduced in this section was extended by O'Meara to integral quadratic forms over local fields.

§ 2. Local representation masses and $\mathbf{Z}_p$-equivalence of forms

We identify $\mathbf{Q}_p$ with its topological dual by defining $<n, m> = \chi_p(nm)$, where χ_p is Tate's character:

$$\chi_p(a) = \exp(2\pi i \sum_{s<0} a_s p^s),$$

if $a = \sum_{s \geqslant s_o} a_s p^s$. Let dn be the Haar measure of $\mathbf{Q}_p$ normalized by $dn(\mathbf{Z}_p) = 1$. As is well-known, dn is selfdual. Let dx be the Haar measure of $\mathbf{Q}_p$ naturally induced by dn.

Let f be a non-singular integral p-adic quadratic form in $k \geqslant 1$ variables. We shall deal in this section with the representation mass function given by (0.1) for $\phi = 1_{(\mathbf{Z}p)^k}$. That is, we define for all $n_o \in \mathbf{Q}_p$:

$$r(n_o, f, \mathbf{Z}_p) = \lim_{U \to \{n_o\}} \left(dx(f^{-1}(U) \cap \mathbf{Z}_p^k)/dnU\right),$$

whenever this limit exists. Clearly r has support contained in $\mathbf{Z}_p$. We can also consider the Gauss-Weil transform of $1_{(\mathbf{Z}p)^k}$ by f given by

$$\theta(m, f, \mathbf{Q}_p) = \int_{\mathbf{Z}_p^k} <f(x), m> dx .$$

The relationship between these representation masses and the ones introduced in the preceeding section is given in the following

Lemma 2.1. i) *Let* $n \in \mathbf{Z}_p, n \neq 0,$ *and* $t > v_p(4n)$. *Then*

$$r(n, f, \mathbf{Z}_p) = \lim_{s \to \infty} p^{(1-k)s} r(n, f, p^s) = p^{(1-k)t} r(n, f, p^t) .$$

ii) *Let* $m \in \mathbf{Z}_p$ *and* $u \in \mathbf{Z}_p, t \geqslant 1$ *be chosen arbitrarily satisfying* $m = up^{-t}$. *Then*

$$\theta(m, f, \mathbf{Q}_p) = p^{-kt}\theta(u, f, p^t) .$$

Proof. i) Let $U_t = n + p^t\mathbf{Z}_p$. We have $dn(U_t) = p^{-t}$ and

$$dx(f^{-1}(U_t)\cap\mathbf{Z}_p^k) = \sum_{a\in(\mathbf{Z}/p^t\mathbf{Z})^k} dx(f^{-1}(U_t)\cap(a+p^t\mathbf{Z}_p^k)) = p^{-kt}r(n, f, p^t),$$

since $f^{-1}(U_t)\cap(a+p^t\mathbf{Z}_p^k)$ is equal to $a + p^t\mathbf{Z}_p^k$ or vacuous, according to $f(a) \equiv n \pmod{p^t}$ or not. This proves the first equality in i).

We want now to show that $p^{(1-k)s}\, r_{p^s}(n) = p^{(1-k)(s-1)}\, r_{p^{s-1}}(n)$, for all $s > t$. We know that

$$r(n, f, p^s) = p^{-s}\sum_{u=1}^{p^s} \theta(u, f, p^s)\exp(-2\pi iunp^{-s}).$$

Let us denote by A and B the sum of the terms satisfying $p \mid u$ and $p \nmid u$, respectively. Clearly $A = p^{k-1}\, r(n, f, p^{s-1})$; hence, we are reduced to proving $B = 0$. Taking into account the explicit computations of Gauss sums (Proposition 1.1), we can express the sum B as

$$B = \begin{cases} C \displaystyle\sum_{u\in(\mathbf{Z}/p^s\mathbf{Z})^*} \left(\frac{u}{p}\right)^a \exp(-2\pi iunp^{-s}) & \text{if} \quad p > 2 \\[2ex] D \displaystyle\sum_{u\in(\mathbf{Z}/2^s\mathbf{Z})^*} \left(\frac{2}{u}\right)^b \exp\left(\frac{2\pi iu}{8}\right)^c \exp(-2\pi iun2^{-s}) & \text{if} \quad p = 2, \end{cases}$$

where C, D, a, b, c depend on f and s, but are independent of u. Now, $\exp(-2\pi inp^{-s})$ is a primitive p^l-th root of 1 with $l > 1$ if $p > 2$, and $l > 3$ if $p = 2$. One can check that, for any function φ defined on $(\mathbf{Z}/p^m Z)^*$, $m \geqslant 1$ and ξ any primitive p^l-th root of 1, $l > m$, one has

$$\sum_{u\in(Z/p^l\mathbf{Z})^*} \varphi(u)\xi^u = 0.$$

In particular, B must be zero.

In order to prove ii) we need only to observe that

$$\theta(m, f, \mathbf{Q}_p) = \int_{\mathbf{Z}_p^k} \exp(2\pi if(x)up^{-t})dx$$

$$= \sum_{a\in(\mathbf{Z}/p^t\mathbf{Z})^k} \exp(2\pi if(a)up^{-t}) \int_{a+p^t\mathbf{Z}_p^k} dx = p^{-kt}\theta(u, f, p^t). \qquad \square$$

Remark. After Siegel [13], it was very well known that for $n \neq 0$ the values $p^{(1-k)t}\, r(n, f, p^t)$ become constant for $t > 2v_p(4n)$. Lemma 2.1 shows that the minimum value of t with this property can be taken equal to half of the one found by Siegel.

By Lemma 2.1, $r(\ ,f,\mathbf{Z}_p)$ is locally constant, hence continuous on $\mathbf{Q}_p^*$, and $r(n,f,\mathbf{Z}_p) = 0$ if and only if n is not represented by f in $\mathbf{Z}_p$. The fundamental fact is that r is integrable on $\mathbf{Z}_p$ and θ is its Fourier transform. This is well-known [4]. For the sake of completeness we give a short proof of this result using only the background introduced up to now.

PROPOSITION 2.2. *$r \in L^1(\mathbf{Z}_p)$ and*

$$\theta(m,f,\mathbf{Q}_p) = \int_{\mathbf{Z}_p} r(n,f,\mathbf{Z}_p) <n,m> dn\,.$$

Proof. We assume $p > 2$. For $p = 2$ the proof works in the same way with minor modifications left to the reader. Let $m = up^{-s}$, $u \in \mathbf{Z}_p$, $s \geqslant 0$. For all $t > s$, $\mathbf{Z}_p \backslash p^t\mathbf{Z}_p$ is compact, hence $r(n)$, being continuous, is integrable and we have by Lemma 2.1:

$$\int_{Z_p\backslash p^t\mathbf{Z}_p} r(n,f,\mathbf{Z}_p) <n,m> dn = \sum_{\substack{a\in Z/p^t\mathbf{Z}\\ a\neq 0}} \int_{a+p^t\mathbf{Z}_p} r(n,f,\mathbf{Z}_p) <n,m> dn$$

$$= \sum_{\substack{a\in Z/p^t\mathbf{Z}\\ a\neq 0}} p^{-kt}\, r(a,f,p^t) \exp(2\pi i a u p^{-s}) = p^{-kt}\big(\theta(p^{t-s}u,f,p^t) - r(0,f,p^t)\big)$$

$$= \theta(m,f,\mathbf{Q}_p) - p^{-kt} r(0,f,p^t)\,.$$

Both assertions of the proposition are consequences of Lebesgue's dominated convergence theorem if $p^{-kt}r(0,f,p^t)$ tends to zero as t tends to infinity. This is checked immediately for $k = 1$. For $k > 1$ it can be easily deduced from (1.1) and the explicit computation of Gauss sums in the preceding section. □

We are ready to prove a crucial fact for the rest of the paper:

THEOREM 2.3. *Let f, g be two non-singular integral p-adic quadratic forms in k variables. If $p = 2$, assume that they are of the same type. The following conditions are equivalent:*

i) $f \sim g$ *over* $\mathbf{Z}_p$,

ii) $r(\ ,f,\mathbf{Z}_p) = r(\ ,g,\mathbf{Z}_p)$,

iii) $\theta(\ ,f,\mathbf{Q}_p) = \theta(\ ,g,\mathbf{Q}_p)$.

Proof. If $f \sim g$ over $\mathbf{Z}_p$, then $f \sim g$ over $\mathbf{Z}/p^t\mathbf{Z}$ and $r(\ ,f,p^t) = r(\ ,g,p^t)$ for all $t \geqslant 1$. By Lemma 2.1 this implies ii). By Proposition 2.2, ii) implies iii). Again by Lemma 2.1, iii) implies that $\theta(\ ,f,p^t) = \theta(\ ,g,p^t)$ for all $t \geqslant 1$, therefore condition i) follows now from Theorem 1.2. □

Let K be a local field and f a non-singular quadratic form in k variables defined over K. If ϕ is a Schwartz-Bruhat function on K^k, the representation mass function $r_\phi(\ ,f,K)$ defined as in (0.1) coincides with another classical representation mass function introduced by Weil. This is Weil's procedure (see [4] for the details): for $n \neq 0$, the $(k-1)$-differential forms

$$\omega_i(x) = (-1)^{i-1}(D_i f)^{-1}\, dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_k\,,$$

induce a gauge form ω_n on the affine variety $f^{-1}(n)$. Since we are in a local field, ω_n induces a positive measure $|\omega_n|$ on $f^{-1}(n)$ such that for every continuous function φ on K^k with compact support not containing zero we have

$$\text{(2.1)} \qquad \int_{K^k} \varphi(x)dx = \int_K \left(\int_{f^{-1}(n)} \varphi\, |\omega_n| \right) dn\,.$$

The representation mass of $n \in K^*$ by f with respect to ϕ is then defined as

$$F_\phi(n) = \int_{f^{-1}(n)} \phi\, |\omega_n|\,.$$

This function is continuous and after (2.1) it is easy to prove that F_ϕ is integrable and its Fourier transform coincides with the Gauss-Weil transform:

$$\int_{K^k} \phi(x) < f(x), m > dx = \int_K F_\phi(n) < n, m > dn\,.$$

Let now $n_o \in K^*$ and let U be any open neighbourhood of n_o. From (2.1) it is also easy to justify that:

$$\int_{f^{-1}(U)} \phi(x)dx = \int_U F_\phi(n)dn\,.$$

Since F_ϕ is continuous and K is locally compact, we have also:

$$F_\phi(n_o) = \lim_{U \to \{n_o\}} \left(\int_U F_\phi(n)dn \Big/ \int_U dn \right) = r_\phi(n_o)\,,$$

thus $F_\phi = r_\phi$ on K^*.