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possible anymore, as the following example shows: By Proposition 1.1 the diagonal forms  $f = \langle 1, 2, 2, 2, 4 \rangle$ ,  $g = \langle 1, 1, 2, 4, 4 \rangle$  have the same Gauss sums  $\theta(\cdot, f, 2^t) = \theta(\cdot, g, 2^t)$  for all  $t \geq 1$ , however they are obviously not  $\mathbf{Z}_2$ -equivalent.

The theory of Minkowski reproduced in this section was extended by O'Meara to integral quadratic forms over local fields.

## § 2. LOCAL REPRESENTATION MASSES AND $\mathbf{Z}_p$ -EQUIVALENCE OF FORMS

We identify  $\mathbf{Q}_p$  with its topological dual by defining  $\langle n, m \rangle = \chi_p(nm)$ , where  $\chi_p$  is Tate's character:

$$\chi_p(a) = \exp\left(2\pi i \sum_{s < 0} a_s p^s\right),$$

if  $a = \sum_{s \geq s_0} a_s p^s$ . Let  $dn$  be the Haar measure of  $\mathbf{Q}_p$  normalized by  $dn(\mathbf{Z}_p) = 1$ . As is well-known,  $dn$  is selfdual. Let  $dx$  be the Haar measure of  $\mathbf{Q}_p$  naturally induced by  $dn$ .

Let  $f$  be a non-singular integral  $p$ -adic quadratic form in  $k \geq 1$  variables. We shall deal in this section with the representation mass function given by (0.1) for  $\phi = 1_{(\mathbf{Z}_p)^k}$ . That is, we define for all  $n_o \in \mathbf{Q}_p$ :

$$r(n_o, f, \mathbf{Z}_p) = \lim_{U \rightarrow \{n_o\}} (dx(f^{-1}(U) \cap \mathbf{Z}_p^k) / dnU),$$

whenever this limit exists. Clearly  $r$  has support contained in  $\mathbf{Z}_p$ . We can also consider the Gauss-Weil transform of  $1_{(\mathbf{Z}_p)^k}$  by  $f$  given by

$$\theta(m, f, \mathbf{Q}_p) = \int_{\mathbf{Z}_p^k} \langle f(x), m \rangle dx.$$

The relationship between these representation masses and the ones introduced in the preceding section is given in the following

LEMMA 2.1. i) Let  $n \in \mathbf{Z}_p$ ,  $n \neq 0$ , and  $t > v_p(4n)$ . Then

$$r(n, f, \mathbf{Z}_p) = \lim_{s \rightarrow \infty} p^{(1-k)s} r(n, f, p^s) = p^{(1-k)t} r(n, f, p^t).$$

ii) Let  $m \in \mathbf{Z}_p$  and  $u \in \mathbf{Z}_p$ ,  $t \geq 1$  be chosen arbitrarily satisfying  $m = up^{-t}$ . Then

$$\theta(m, f, \mathbf{Q}_p) = p^{-kt} \theta(u, f, p^t).$$

*Proof.* i) Let  $U_t = n + p^t \mathbf{Z}_p$ . We have  $dn(U_t) = p^{-t}$  and

$$dx(f^{-1}(U_t) \cap \mathbf{Z}_p^k) = \sum_{a \in (\mathbf{Z}/p^t \mathbf{Z})^k} dx(f^{-1}(U_t) \cap (a + p^t \mathbf{Z}_p^k)) = p^{-kt} r(n, f, p^t),$$

since  $f^{-1}(U_t) \cap (a + p^t \mathbf{Z}_p^k)$  is equal to  $a + p^t \mathbf{Z}_p^k$  or vacuous, according to  $f(a) \equiv n \pmod{p^t}$  or not. This proves the first equality in i).

We want now to show that  $p^{(1-k)s} r_{p^s}(n) = p^{(1-k)(s-1)} r_{p^{s-1}}(n)$ , for all  $s > t$ . We know that

$$r(n, f, p^s) = p^{-s} \sum_{u=1}^{p^s} \theta(u, f, p^s) \exp(-2\pi i u n p^{-s}).$$

Let us denote by  $A$  and  $B$  the sum of the terms satisfying  $p \mid u$  and  $p \nmid u$ , respectively. Clearly  $A = p^{k-1} r(n, f, p^{s-1})$ ; hence, we are reduced to proving  $B = 0$ . Taking into account the explicit computations of Gauss sums (Proposition 1.1), we can express the sum  $B$  as

$$B = \begin{cases} C \sum_{u \in (\mathbf{Z}/p^s \mathbf{Z})^*} \left(\frac{u}{p}\right)^a \exp(-2\pi i u n p^{-s}) & \text{if } p > 2 \\ D \sum_{u \in (\mathbf{Z}/2^s \mathbf{Z})^*} \left(\frac{2}{u}\right)^b \exp\left(\frac{2\pi i u}{8}\right)^c \exp(-2\pi i u n 2^{-s}) & \text{if } p = 2, \end{cases}$$

where  $C, D, a, b, c$  depend on  $f$  and  $s$ , but are independent of  $u$ . Now,  $\exp(-2\pi i u n p^{-s})$  is a primitive  $p^l$ -th root of 1 with  $l > 1$  if  $p > 2$ , and  $l > 3$  if  $p = 2$ . One can check that, for any function  $\varphi$  defined on  $(\mathbf{Z}/p^m \mathbf{Z})^*$ ,  $m \geq 1$  and  $\xi$  any primitive  $p^l$ -th root of 1,  $l > m$ , one has

$$\sum_{u \in (\mathbf{Z}/p^l \mathbf{Z})^*} \varphi(u) \xi^u = 0.$$

In particular,  $B$  must be zero.

In order to prove ii) we need only to observe that

$$\begin{aligned} \theta(m, f, \mathbf{Q}_p) &= \int_{\mathbf{Z}_p^k} \exp(2\pi i f(x) u p^{-t}) dx \\ &= \sum_{a \in (\mathbf{Z}/p^t \mathbf{Z})^k} \exp(2\pi i f(a) u p^{-t}) \int_{a + p^t \mathbf{Z}_p^k} dx = p^{-kt} \theta(u, f, p^t). \quad \square \end{aligned}$$

*Remark.* After Siegel [13], it was very well known that for  $n \neq 0$  the values  $p^{(1-k)t} r(n, f, p^t)$  become constant for  $t > 2v_p(4n)$ . Lemma 2.1 shows that the minimum value of  $t$  with this property can be taken equal to half of the one found by Siegel.

By Lemma 2.1,  $r(\cdot, f, \mathbf{Z}_p)$  is locally constant, hence continuous on  $\mathbf{Q}_p^*$ , and  $r(n, f, \mathbf{Z}_p) = 0$  if and only if  $n$  is not represented by  $f$  in  $\mathbf{Z}_p$ . The fundamental fact is that  $r$  is integrable on  $\mathbf{Z}_p$  and  $\theta$  is its Fourier transform. This is well-known [4]. For the sake of completeness we give a short proof of this result using only the background introduced up to now.

PROPOSITION 2.2.  $r \in L^1(\mathbf{Z}_p)$  and

$$\theta(m, f, \mathbf{Q}_p) = \int_{\mathbf{Z}_p} r(n, f, \mathbf{Z}_p) \langle n, m \rangle dn.$$

*Proof.* We assume  $p > 2$ . For  $p = 2$  the proof works in the same way with minor modifications left to the reader. Let  $m = up^{-s}$ ,  $u \in \mathbf{Z}_p$ ,  $s \geq 0$ . For all  $t > s$ ,  $\mathbf{Z}_p \setminus p^t \mathbf{Z}_p$  is compact, hence  $r(n)$ , being continuous, is integrable and we have by Lemma 2.1:

$$\begin{aligned} \int_{\mathbf{Z}_p \setminus p^t \mathbf{Z}_p} r(n, f, \mathbf{Z}_p) \langle n, m \rangle dn &= \sum_{\substack{a \in \mathbf{Z}/p^t \mathbf{Z} \\ a \neq 0}} \int_{a + p^t \mathbf{Z}_p} r(n, f, \mathbf{Z}_p) \langle n, m \rangle dn \\ &= \sum_{\substack{a \in \mathbf{Z}/p^t \mathbf{Z} \\ a \neq 0}} p^{-kt} r(a, f, p^t) \exp(2\pi i a u p^{-s}) = p^{-kt} (\theta(p^{t-s} u, f, p^t) - r(0, f, p^t)) \\ &= \theta(m, f, \mathbf{Q}_p) - p^{-kt} r(0, f, p^t). \end{aligned}$$

Both assertions of the proposition are consequences of Lebesgue's dominated convergence theorem if  $p^{-kt} r(0, f, p^t)$  tends to zero as  $t$  tends to infinity. This is checked immediately for  $k = 1$ . For  $k > 1$  it can be easily deduced from (1.1) and the explicit computation of Gauss sums in the preceding section.  $\square$

We are ready to prove a crucial fact for the rest of the paper:

THEOREM 2.3. Let  $f, g$  be two non-singular integral  $p$ -adic quadratic forms in  $k$  variables. If  $p = 2$ , assume that they are of the same type. The following conditions are equivalent:

- i)  $f \sim g$  over  $\mathbf{Z}_p$ ,
- ii)  $r(\cdot, f, \mathbf{Z}_p) = r(\cdot, g, \mathbf{Z}_p)$ ,
- iii)  $\theta(\cdot, f, \mathbf{Q}_p) = \theta(\cdot, g, \mathbf{Q}_p)$ .

*Proof.* If  $f \sim g$  over  $\mathbf{Z}_p$ , then  $f \sim g$  over  $\mathbf{Z}/p^t \mathbf{Z}$  and  $r(\cdot, f, p^t) = r(\cdot, g, p^t)$  for all  $t \geq 1$ . By Lemma 2.1 this implies ii). By Proposition 2.2, ii) implies iii). Again by Lemma 2.1, iii) implies that  $\theta(\cdot, f, p^t) = \theta(\cdot, g, p^t)$  for all  $t \geq 1$ , therefore condition i) follows now from Theorem 1.2.  $\square$

Let  $K$  be a local field and  $f$  a non-singular quadratic form in  $k$  variables defined over  $K$ . If  $\phi$  is a Schwartz-Bruhat function on  $K^k$ , the representation mass function  $r_\phi(\cdot, f, K)$  defined as in (0.1) coincides with another classical representation mass function introduced by Weil. This is Weil's procedure (see [4] for the details): for  $n \neq 0$ , the  $(k-1)$ -differential forms

$$\omega_i(x) = (-1)^{i-1} (D_i f)^{-1} dx_1 \wedge \dots \wedge d\hat{x}_i \wedge \dots \wedge dx_k,$$

induce a gauge form  $\omega_n$  on the affine variety  $f^{-1}(n)$ . Since we are in a local field,  $\omega_n$  induces a positive measure  $|\omega_n|$  on  $f^{-1}(n)$  such that for every continuous function  $\phi$  on  $K^k$  with compact support not containing zero we have

$$(2.1) \quad \int_{K^k} \phi(x) dx = \int_K \left( \int_{f^{-1}(n)} \phi |\omega_n| \right) dn.$$

The representation mass of  $n \in K^*$  by  $f$  with respect to  $\phi$  is then defined as

$$F_\phi(n) = \int_{f^{-1}(n)} \phi |\omega_n|.$$

This function is continuous and after (2.1) it is easy to prove that  $F_\phi$  is integrable and its Fourier transform coincides with the Gauss-Weil transform:

$$\int_{K^k} \phi(x) \langle f(x), m \rangle dx = \int_K F_\phi(n) \langle n, m \rangle dn.$$

Let now  $n_o \in K^*$  and let  $U$  be any open neighbourhood of  $n_o$ . From (2.1) it is also easy to justify that:

$$\int_{f^{-1}(U)} \phi(x) dx = \int_U F_\phi(n) dn.$$

Since  $F_\phi$  is continuous and  $K$  is locally compact, we have also:

$$F_\phi(n_o) = \lim_{U \rightarrow \{n_o\}} \left( \int_U F_\phi(n) dn / \int_U dn \right) = r_\phi(n_o),$$

thus  $F_\phi = r_\phi$  on  $K^*$ .