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§ 1. GAUSS SUMS AND EQUIVALENCE OF QUADRATIC FORMS

We summarize in this section some classical criteria, essentially due to Minkowski (cf. [8]), for \mathbf{Z}_p -equivalence of quadratic forms in terms of Gauss sums.

In general, if f and g are two integral quadratic forms in k variables over a ring Λ , and A and B are the symmetric matrices with entries in Λ such that $f(x) = x^T A x$, $g(x) = x^T B x$, we will say that f and g are Λ -equivalent, (resp. of the same Λ -type) if there exist $P, Q \in GL(k, \Lambda)$ such that $B = P^T A P$ (resp. $B = Q A P$). In the first case we shall write " $f \sim g$, over Λ ".

Let p be a prime, $t \geq 1$ an integer and let $\Lambda = \mathbf{Z}/p^t \mathbf{Z}$ with discrete topology. Let dn be the Haar measure of Λ normalized by $dn(\Lambda) = p^t$ and take $\phi = 1$. The representation mass (0.1) of $n \in \Lambda$ by a quadratic form f over Λ is the ordinary number of representations

$$r(n, f, p^t) := \# f^{-1}(n).$$

Its Fourier transform is given by

$$\theta(m, f, p^t) := \sum_{n=1}^{p^t} r(n, f, p^t) \exp(2\pi i n m p^{-t}).$$

It clearly coincides with the Gauss-Weil transform (0.3), which in this case is the ordinary Gauss sum:

$$\theta(m, f, p^t) = \sum_{x \in \Lambda^k} \exp(2\pi i m f(x) p^{-t}).$$

By the Fourier inversion formula we have, moreover,

$$(1.1) \quad r(n, f, p^t) = p^{-t} \sum_{m=1}^{p^t} \theta(m, f, p^t) \exp(-2\pi i m n p^{-t}).$$

As is well known, any integral p -adic form is \mathbf{Z}_p -equivalent to an orthogonal sum of 1-dimensional forms if $p > 2$, and 1-dimensional and 2-dimensional forms if $p = 2$. Since, on the other hand, given two integral p -adic forms f and g we have for every $t \geq 1$

$$\theta(\cdot, f \perp g, p^t) = \theta(\cdot, f, p^t) \theta(\cdot, g, p^t),$$

the θ values of f can be deduced from the next proposition.

PROPOSITION 1.1. i) Let $u, v \in \mathbf{Z}_p, p \nmid uv$ and $s, t \in \mathbf{Z}, s \geq 0, t \geq 1$.
Then

$$\theta(u, p^s v X^2, p^t) = \begin{cases} p^t & \text{if } t \leq s \\ p^{(t+s)/2} \left(\frac{uv}{p}\right)^{t+s} \varepsilon_p^{(t+s)^2} & \text{if } t > s, p > 2 \\ 0 & \text{if } t = s + 1, p = 2 \\ 2^{(t+s+1)/2} \left(\frac{2}{uv}\right)^{t+s+1} \exp(2\pi i uv/8) & \text{if } t > s + 1, p = 2. \end{cases}$$

where $\varepsilon_p = 1$ or i , according to $p \equiv 1$ or $3 \pmod{4}$.

ii) Let $F(X, Y) = vX^2 + 2wXY + zY^2, 2 \nmid (v, w, z)$ be a 2-adic non-diagonalizable integral quadratic form. Then if $t \geq 1$ and $u \in \mathbf{Z}_2$ is odd

$$\theta(u, 2^s F, 2^t) = \begin{cases} 2^{2t} & \text{if } t \leq s. \\ 2^{t+s+1} \left(\frac{2}{d}\right)^{t+s+1} & \text{if } t > s, \end{cases}$$

where $d = vz - w^2$.

Proof. From the definition of θ it is clear that

$$\theta(u, p^s v f, p^t) = \theta(p^s uv, f, p^t) = \begin{cases} p^{tk} & \text{if } t \leq s, \\ p^{sk} \theta(uv, f, p^{t-s}) & \text{if } t > s, \end{cases}$$

for any integral p -adic form f and u, v, s, t as in i). Hence the assertion of i) follows from the well-known values of the Gauss sums $\theta(\cdot, X^2, p^t)$ (cf. [3], Ch. 7, Thms. 5.6 and 5.7).

Let $F(X, Y)$ be as in ii). Being primitive, F is diagonalizable if and only if it represents some odd integer, and this is equivalent to v or z being odd. Suppose that $t > s$ and v and z even. One computes easily by hand that

$$\theta(u, F, 2) = 4, \quad \theta(u, F, 4) = 8 \left(\frac{2}{d}\right).$$

If $t \geq 3$, we get ii) from the equality

$$\theta(u, F, 2^t) = 4\theta(u, F, 2^{t-2}). \quad \square$$

THEOREM 1.2. *Let f, g be two non-singular integral p -adic quadratic forms in k variables. If $p = 2$, assume that they are of the same type. The following conditions are equivalent:*

- i) $f \sim g$ over \mathbf{Z}_p ,
- ii) $r(\cdot, f, p^t) = r(\cdot, g, p^t)$ for all $t \geq 1$,
- iii) $\theta(\cdot, f, p^t) = \theta(\cdot, g, p^t)$ for all $t \geq 1$.

Two \mathbf{Z}_p -equivalent forms are, in particular, $\mathbf{Z}/p^t\mathbf{Z}$ -equivalent for all $t \geq 1$, hence they have the same representation numbers $r(n, f, p^t)$ for all $t \geq 1, n \in \mathbf{Z}_p$. Since $r(\cdot, f, p^t)$ and $\theta(\cdot, f, p^t)$ are Fourier transforms over $\mathbf{Z}/p^t\mathbf{Z}$ one of each other, ii) and iii) are clearly equivalent. Therefore, the proof of Theorem 1.2 is reduced to showing that Gauss sums determine \mathbf{Z}_p -equivalence. This is easy if $p > 2$:

Proof of Theorem 1.2 for $p > 2$. We proceed by induction on k . Let $f(X) = p^s v X^2, g(X) = p^{s'} v' X^2, p \nmid vv'$. By Proposition 1.1, the equality $\theta(1, f, p^t) = \theta(1, g, p^t)$ for $t = s + 1, s + 2$ implies that $s = s'$ and $\left(\frac{v}{p}\right) = \left(\frac{v'}{p}\right)$, thus $f \sim g$ over \mathbf{Z}_p . Let $f = p^s f_0, g = p^{s'} g_0$ be two forms in k variables with f_0, g_0 primitive. If they have the same Gauss sums, then $s = s'$, otherwise, if $s < s'$ by Proposition 1.1 we would have

$$|\theta(1, f, p^{s'})| < \theta(1, g, p^{s'}) = p^{s'k},$$

a contradiction. Since f_0 and g_0 will have the same Gauss sums, we can suppose that f and g are both primitive. Let u be a p -adic unit represented by f and g . It is well known that, over \mathbf{Z}_p , we have splittings

$$f \sim \langle u \rangle \perp f_1, \quad g \sim \langle u \rangle \perp \langle g_1 \rangle.$$

Since $\theta(\cdot, uX^2, p^t)$ never vanishes and \mathbf{Z}_p -equivalent forms have the same Gauss sums, we will have

$$\theta(\cdot, f_1, p^t) = \frac{\theta(\cdot, f, p^t)}{\theta(\cdot, uX^2, p^t)} = \frac{\theta(\cdot, g, p^t)}{\theta(\cdot, uX^2, p^t)} = \theta(\cdot, g_1, p^t),$$

for all t . By the induction hypothesis this implies $f_1 \sim g_1$, hence $f \sim g$ over \mathbf{Z}_p . \square

The proof of Theorem 1.2 for $p = 2$ is much more delicate, due to the fact that Gauss sums can vanish in this case. We need a few properties of 2-adic forms which we sum up in Lemma 1.3 below.

We recall that a primitive 2-adic integral quadratic form is called *properly primitive* if it represents some odd integer, otherwise it is called *improperly primitive*. Clearly a 2-dimensional primitive form is properly primitive if and only if it is diagonalizable over \mathbf{Z}_2 .

LEMMA 1.3. (cf. [1, Ch. 8]). Let $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $H' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Then

i) Every improperly primitive form over \mathbf{Z}_2 is \mathbf{Z}_2 -equivalent to one of the following two:

$$H \perp \dots \perp H \perp H \quad \text{or} \quad H \perp \dots \perp H \perp H'.$$

ii) For any 2-adic unit u we have splittings over \mathbf{Z}_2 :

$$\langle u \rangle \perp H \sim \langle u, 1, -1 \rangle,$$

$$\langle u \rangle \perp H' \sim \langle u-2, u+2, (2u+3)(u+2)^{-1} \rangle,$$

$$\langle 2u \rangle \perp H \sim \langle 2u+8 \rangle \perp H'.$$

Proof of Theorem 1.2 for $p = 2$. By induction on k . Let $f(X) = 2^s v X^2$, $g(X) = 2^{s'} v' X^2$, $2 \nmid vv'$. By Proposition 1.1, $\theta(1, f, 2^t) = \theta(1, g, 2^t)$ for $t = s + 2$, $s + 3$ implies that $s = s'$ and $v \equiv v' \pmod{8}$, hence $f \sim g$ over \mathbf{Z}_2 . Let f and g be two forms in k variables, $k \geq 2$, of the same type. We consider the splittings over \mathbf{Z}_2 :

$$f \sim 2^{s_1} f_1 \perp \dots \perp 2^{s_r} f_r,$$

$$g \sim 2^{s_1} g_1 \perp \dots \perp 2^{s_r} g_r, \quad 0 \leq s_1 < s_2 < \dots < s_r,$$

f_i, g_i with unit determinant and the same number of variables, k_i , for all i . Without restriction we can suppose that f and g are primitive, that is $s_1 = 0$. If f and g have the same Gauss sums, then for each i , f_i and g_i are both properly or improperly primitive since, by Proposition 1.1, this is equivalent to the vanishing or not of $\theta(1, f, 2^{s_i+1})$. The proof proceeds in a different way according to whether f_1, g_1 are properly or improperly primitive.

Suppose that f_1 and g_1 are improperly primitive. If $k_1 > 2$, by i) of Lemma 1.3 we have, over \mathbf{Z}_2 ,

$$f \sim H \perp F, \quad g \sim H \perp G,$$

and, since $\theta(\cdot, H, 2^t)$ never vanishes, we have

$$\theta(\cdot, F, 2^t) = \frac{\theta(\cdot, f, 2^t)}{\theta(\cdot, H, 2^t)} = \theta(\cdot, G, 2^t),$$

for all t . By the induction hypothesis $F \sim G$, hence $f \sim g$ over \mathbf{Z}_2 . If $k_1 = 2$ and $f_1 \sim g_1$ over \mathbf{Z}_2 , we can proceed as above. Suppose that $k_1 = 2$ and

$$\begin{aligned} f &\sim H \perp 2^{s_2} f_2 \perp \dots \perp 2^{s_r} f_r, \\ g &\sim H' \perp 2^{s_2} g_2 \perp \dots \perp 2^{s_r} g_r. \end{aligned}$$

If $s_2 > 1$ (or $k = k_1 = 2$) or f_2, g_2 are improperly primitive we have

$$\theta(1, f, 4) = 2^{3+2(k-2)} = -\theta(1, g, 4),$$

a contradiction. Hence $s_2 = 1$ and f_2, g_2 are diagonalizable. By ii) of Lemma 1.3, $g \sim H \perp 2^{s_2} g'_2 \perp \dots$, over \mathbf{Z}_2 , and we can proceed as above.

Suppose now that f_1 and g_1 are properly primitive. Let u be a 2-adic unit represented by f and g . We have splittings over \mathbf{Z}_2 :

$$(1.2) \quad f \sim \langle u \rangle \perp F, \quad g \sim \langle u \rangle \perp G.$$

Since $\theta(\cdot, uX^2, 2^t) \neq 0$ for $t \neq 1$, we get $\theta(\cdot, F, 2^t) = \theta(\cdot, G, 2^t)$ for all $t \neq 1$. We have only to prove that $\theta(\cdot, F, 2) = \theta(\cdot, G, 2)$ and the claim will follow from the induction hypothesis. If $k_1 = 1$ or F and G are both properly or improperly primitive we are done. Assume that F is properly and G improperly primitive. This is possible indeed (see ii) of Lemma 1.3). By ii) of Lemma 1.3 we can always find a \mathbf{Z}_2 -splitting $g \sim \langle u \rangle \perp G'$, with G' properly primitive except for the case that over \mathbf{Z}_2

$$g \sim \langle u \rangle \perp H' \perp 2^{s_2} g_2 \perp \dots,$$

with $k_1 = 3$ and $s_2 > 1$ (or $k = 3$) or g_2 improperly primitive. Let us assume in this case that over \mathbf{Z}_2

$$f \sim \langle u, v, w \rangle \perp 2^{s_2} f_2 \perp \dots.$$

From $\theta(1, f, 4) = \theta(1, g, 4)$ we get

$$\exp(2\pi i(v+w)/8) = -\left(\frac{2}{vw}\right),$$

or, equivalently, $vw \equiv 3 \pmod{8}$. This implies that either v or w are congruent (mod 8) to any of $u - 2, u + 2$; hence, changing u by v or w we get a splitting (1.2) with F and G both properly primitive. \square

Remark. For $p = 2$ and $k \leq 4$ we could remove in the theorem the condition of f and g being of the same type. For $k \geq 5$ this is not

possible anymore, as the following example shows: By Proposition 1.1 the diagonal forms $f = \langle 1, 2, 2, 2, 4 \rangle$, $g = \langle 1, 1, 2, 4, 4 \rangle$ have the same Gauss sums $\theta(\cdot, f, 2^t) = \theta(\cdot, g, 2^t)$ for all $t \geq 1$, however they are obviously not \mathbf{Z}_2 -equivalent.

The theory of Minkowski reproduced in this section was extended by O'Meara to integral quadratic forms over local fields.

§ 2. LOCAL REPRESENTATION MASSES AND \mathbf{Z}_p -EQUIVALENCE OF FORMS

We identify \mathbf{Q}_p with its topological dual by defining $\langle n, m \rangle = \chi_p(nm)$, where χ_p is Tate's character:

$$\chi_p(a) = \exp\left(2\pi i \sum_{s < 0} a_s p^s\right),$$

if $a = \sum_{s \geq s_0} a_s p^s$. Let dn be the Haar measure of \mathbf{Q}_p normalized by $dn(\mathbf{Z}_p) = 1$. As is well-known, dn is selfdual. Let dx be the Haar measure of \mathbf{Q}_p naturally induced by dn .

Let f be a non-singular integral p -adic quadratic form in $k \geq 1$ variables. We shall deal in this section with the representation mass function given by (0.1) for $\phi = 1_{(\mathbf{Z}_p)^k}$. That is, we define for all $n_o \in \mathbf{Q}_p$:

$$r(n_o, f, \mathbf{Z}_p) = \lim_{U \rightarrow \{n_o\}} (dx(f^{-1}(U) \cap \mathbf{Z}_p^k) / dnU),$$

whenever this limit exists. Clearly r has support contained in \mathbf{Z}_p . We can also consider the Gauss-Weil transform of $1_{(\mathbf{Z}_p)^k}$ by f given by

$$\theta(m, f, \mathbf{Q}_p) = \int_{\mathbf{Z}_p^k} \langle f(x), m \rangle dx.$$

The relationship between these representation masses and the ones introduced in the preceding section is given in the following

LEMMA 2.1. i) Let $n \in \mathbf{Z}_p$, $n \neq 0$, and $t > v_p(4n)$. Then

$$r(n, f, \mathbf{Z}_p) = \lim_{s \rightarrow \infty} p^{(1-k)s} r(n, f, p^s) = p^{(1-k)t} r(n, f, p^t).$$

ii) Let $m \in \mathbf{Z}_p$ and $u \in \mathbf{Z}_p$, $t \geq 1$ be chosen arbitrarily satisfying $m = up^{-t}$. Then

$$\theta(m, f, \mathbf{Q}_p) = p^{-kt} \theta(u, f, p^t).$$