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ZETA FUNCTIONS AND GENUS OF QUADRATIC FORMS

by Pilar BAYER and Enric NART [1])

Let f be a positive definite **Z**-integral quadratic form in k variables and let $r(n, f)$ be the number of integral representations of an integer n by f. A classical analytical invariant associated to f is Epstein's zeta function $\zeta(s, f) = \sum_{n=1}^{\infty} r(n, f)n^{-s}$, for a complex s with $\operatorname{Re} s > k/2$. In the indefinite case, the number of integral representations of an integer by a form can be infinite. Nevertheless, to any **Z**-integral quadratic form f and integer n, Siegel attached representation masses $r(n, f)$ and studied the corresponding complex zeta function $\zeta(s, f)$.

It is a natural question to ask to what extent $\zeta(s, f)$ contains information about the arithmetic properties of f. It should be noted that it doesn't determine the equivalence class of f, since examples of inequivalent forms over **Z** with the same representation numbers were already pointed out by Witt, Kneser and Kitaoka [19, 7, 5]. However, it was conjectured by Kitaoka in [5] that for f even and positive definite, $\zeta(s, f)$ determines the genus of f. We show in Theorem 5.2 that if two non-singular, **Z**-integral quadratic forms, with the same signature and the same 2-type, have the same representation masses, then they must belong to the same genus. From the analytical point of view, this is equivalent to the statement that two forms, as above, with the same complex zeta function must be "isogenous".

The proof of this theorem relies on a previous study of representation masses of integral quadratic forms, in the local case.

Given a locally compact ring Λ and a non singular Λ-integral quadratic form f in k variables, there are several ways to attach to f representation masses. Most of them can be unified by the following definition:

$$(0.1) \qquad r_{\phi}(n_0, f, \Lambda) = \lim_{U \to \{n_0\}} \left(\int_{f^{-1}(U)} \phi(x) dx \Big/ \int_U dn \right),$$

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for $n_0 \in \Lambda$. Here dn denotes a Haar measure on $(\Lambda, +)$, dx the Haar measure naturally induced on Λ^k by dn, and ϕ is a function in $L^1(\Lambda^k)$. Let $\hat{\Lambda}$ be the Pontrjagin dual of Λ.

In all cases we shall deal with, r_ϕ will be integrable over Λ. This makes possible the definition of local theta functions simply by taking Fourier transforms:

$$(0.2) \qquad \theta_\phi(m, f, \hat{\Lambda}) := \int_\Lambda r_\phi(n, f, \Lambda) < n, m > dn\,.$$

Since

$$(0.3) \qquad \int_\Lambda r_\phi(n, f, \Lambda) < n, m > dn = \int_{\Lambda^k} \phi(x) < f(x), m > dx\,,$$

these theta functions are nothing else but the Gauss-Weil transform of ϕ by f.

If $\Lambda = \mathbf{Z}/p^t\mathbf{Z}$, the representation masses (0.1) are the congruential representation numbers of the quadratic form and the Gauss-Weil transform (0.3) yields the ordinary Gauss sums attached to f. They were studied by Minkowski in his important paper [8]; his results are reproduced in § 1.

In general, it is clear from the definition that the representation masses $r_\phi(\ , f, \Lambda)$ are invariant of the Λ-equivalence class of f. It is a crucial fact that in favorable cases they are, in fact, a complete system of invariants of such a class. This is worked out in Sections 2, 3 and 4 for $\Lambda = \mathbf{Z}_p$, $\mathbf{Q}_p$ and $\mathbf{A}$; the core of the proof is always the same: Λ-equivalence is determined by the theta function $\theta_\phi(\ , f, \hat{\Lambda})$, for a suitable ϕ. The corresponding global statement is then obtained in Section 5 by means of Siegel's formula and the local result for $\mathbf{Z}_p$. We prove first in Theorem 5.1 that two forms with the same average-in-the-genus theta series must belong to the same genus, if they have the same signature and the same 2-type. This result was anticipated by Siegel [13, page 374] for definite forms in five or more variables. Our proof, essentially of a local nature, deals simultaneously with the definite case (and $k \geq 3$) and the indefinite case (and $k \geq 4$). From Theorem 5.1 we deduce Theorem 5.2 showing that two forms, as above, with the same representation masses must have also the same average-in-the-genus masses.

§ 1. Gauss sums and equivalence of quadratic forms

We summarize in this section some classical criteria, essentially due to Minskowski (cf. [8]), for $\mathbf{Z}_p$-equivalence of quadratic forms in terms of Gauss sums.

In general, if f and g are two integral quadratic forms in k variables over a ring Λ, and A and B are the symmetric matrices with entries in Λ such that $f(x) = x^T A x$, $g(x) = x^T B x$, we will say that f and g are Λ-*equivalent*, (resp. of the *same* Λ-*type*) if there exist $P, Q \in GL(k, \Lambda)$ such that $B = P^T A P$ (resp. $B = QAP$). In the first case we shall write "$f \sim g$, over Λ".

Let p be a prime, $t \geqslant 1$ an integer and let $\Lambda = \mathbf{Z}/p^t\mathbf{Z}$ with discrete topology. Let dn be the Haar measure of Λ normalized by $dn(\Lambda) = p^t$ and take $\phi = 1$. The representation mass (0.1) of $n \in \Lambda$ by a quadratic form f over Λ is the ordinary number of representations

$$r(n, f, p^t) := \# f^{-1}(n)\,.$$

Its Fourier transform is given by

$$\theta(m, f, p^t) := \sum_{n=1}^{p^t} r(n, f, p^t) \exp(2\pi i n m p^{-t})\,.$$

It clearly coincides with the Gauss-Weil transform (0.3), which in this case is the ordinary Gauss sum:

$$\theta(m, f, p^t) = \sum_{x \in \Lambda^k} \exp\left(2\pi i m f(x) p^{-t}\right).$$

By the Fourier inversion formula we have, moreover,

$$(1.1) \qquad r(n, f, p^t) = p^{-t} \sum_{m=1}^{p^t} \theta(m, f, p^t) \exp(-2\pi i m n p^{-t})\,.$$

As is well known, any integral p-adic form is $\mathbf{Z}_p$-equivalent to an orthogonal sum of 1-dimensional forms if $p > 2$, and 1-dimensional and 2-dimensional forms if $p = 2$. Since, on the other hand, given two integral p-adic forms f and g we have for every $t \geqslant 1$

$$\theta(\ , f \perp g, p^t) = \theta(\ , f, p^t)\, \theta(\ , g, p^t)\,,$$

the θ values of f can be deduced from the next proposition.

PROPOSITION 1.1. i) *Let* $u, v \in \mathbf{Z}_p, p \nmid uv$ *and* $s, t \in \mathbf{Z}, s \geqslant 0, t \geqslant 1$. *Then*

$$\theta(u, p^s vX^2, p^t) = \begin{cases} p^t & \text{if } t \leqslant s \\ p^{(t+s)/2}\left(\dfrac{uv}{p}\right)^{t+s} \varepsilon_p^{(t+s)^2} & \text{if } t > s, p > 2 \\ 0 & \text{if } t = s+1, p = 2 \\ 2^{(t+s+1)/2}\left(\dfrac{2}{uv}\right)^{t+s+1} \exp(2\pi i uv/8) & \text{if } t > s+1, p = 2 . \end{cases}$$

where $\varepsilon_p = 1$ *or* i, *according to* $p \equiv 1$ *or* $3 \pmod 4$.

ii) *Let* $F(X, Y) = vX^2 + 2wXY + zY^2, 2 \nmid (v, w, z)$ *be a 2-adic non-diagonalizable integral quadratic form. Then if* $t \geqslant 1$ *and* $u \in \mathbf{Z}_2$ *is odd*

$$\theta(u, 2^s F, 2^t) = \begin{cases} 2^{2t} & \text{if } t \leqslant s . \\ 2^{t+s+1}\left(\dfrac{2}{d}\right)^{t+s+1} & \text{if } t > s , \end{cases}$$

where $d = vz - w^2$.

Proof. From the definition of θ it is clear that

$$\theta(u, p^s vf, p^t) = \theta(p^s uv, f, p^t) = \begin{cases} p^{tk} & \text{if } t \leqslant s , \\ p^{sk}\theta(uv, f, p^{t-s}) & \text{if } t > s , \end{cases}$$

for any integral p-adic form f and u, v, s, t as in i). Hence the assertion of i) follows from the well-known values of the Gauss sums $\theta(\, , X^2, p^t)$ (cf. [3], Ch. 7, Thms. 5.6 and 5.7).

Let $F(X, Y)$ be as in ii). Being primitive, F is diagonalizable if and only if it represents some odd integer, and this is equivalent to v or z being odd. Suppose that $t > s$ and v and z even. One computes easily by hand that

$$\theta(u, F, 2) = 4 , \quad \theta(u, F, 4) = 8\left(\frac{2}{d}\right) .$$

If $t \geqslant 3$, we get ii) from the equality

$$\theta(u, F, 2^t) = 4\theta(u, F, 2^{t-2}) . \qquad \square$$

THEOREM 1.2. *Let* f, g *be two non-singular integral p-adic quadratic forms in* k *variables. If* $p = 2$, *assume that they are of the same type. The following conditions are equivalent:*

i) $f \sim g$ *over* $\mathbf{Z}_p$,

ii) $r(\ , f, p^t) = r(\ , g, p^t)$ *for all* $t \geqslant 1$,

iii) $\theta(\ , f, p^t) = \theta(\ , g, p^t)$ *for all* $t \geqslant 1$.

Two $\mathbf{Z}_p$-equivalent forms are, in particular, $\mathbf{Z}/p^t\mathbf{Z}$-equivalent for all $t \geqslant 1$, hence they have the same representation numbers $r(n, f, p^t)$ for all $t \geqslant 1$, $n \in \mathbf{Z}_p$. Since $r(\ , f, p^t)$ and $\theta(\ , f, p^t)$ are Fourier transforms over $\mathbf{Z}/p^t\mathbf{Z}$ one of each other, ii) and iii) are clearly equivalent. Therefore, the proof of Theorem 1.2 is reduced to showing that Gauss sums determine $\mathbf{Z}_p$-equivalence. This is easy if $p > 2$:

Proof of Theorem 1.2 for $p > 2$. We proceed by induction on k. Let $f(X) = p^s v X^2$, $g(X) = p^{s'} v' X^2$, $p \nmid vv'$. By Proposition 1.1, the equality $\theta(1, f, p^t) = \theta(1, g, p^t)$ for $t = s + 1, s + 2$ implies that $s = s'$ and $\left(\frac{v}{p}\right) = \left(\frac{v'}{p}\right)$, thus $f \sim g$ over $\mathbf{Z}_p$. Let $f = p^s f_0$, $g = p^{s'} g_0$ be two forms in k variables with f_0, g_0 primitive. If they have the same Gauss sums, then $s = s'$, otherwise, if $s < s'$ by Proposition 1.1 we would have

$$| \theta(1, f, p^{s'}) | < \theta(1, g, p^{s'}) = p^{s'k},$$

a contradiction. Since f_0 and g_0 will have the same Gauss sums, we can suppose that f and g are both primitive. Let u be a p-adic unit represented by f and g. It is well known that, over $\mathbf{Z}_p$, we have splittings

$$f \sim \langle u \rangle \perp f_1 , \quad g \sim \langle u \rangle \perp \langle g_1 \rangle .$$

Since $\theta(\ , uX^2, p^t)$ never vanishes and $\mathbf{Z}_p$-equivalent forms have the same Gauss sums, we will have

$$\theta(\ , f_1, p^t) = \frac{\theta(\ , f, p^t)}{\theta(\ , uX^2, p^t)} = \frac{\theta(\ , g, p^t)}{\theta(\ , uX^2, p^t)} = \theta(\ , g_1, p^t),$$

for all t. By the induction hypothesis this implies $f_1 \sim g_1$, hence $f \sim g$ over $\mathbf{Z}_p$. □

The proof of Theorem 1.2 for $p = 2$ is much more delicate, due to the fact that Gauss sums can vanish in this case. We need a few properties of 2-adic forms which we sum up in Lemma 1.3 below.

We recall that a primitive 2-adic integral quadratic form is called *properly primitive* if it represents some odd integer, otherwise it is called *improperly primitive.* Clearly a 2-dimensional primitive form is properly primitive if and only if it is diagonalizable over $\mathbf{Z}_2$.

LEMMA 1.3. (cf. [1, Ch. 8]). *Let* $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $H' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. *Then*

i) *Every improperly primitive form over* $\mathbf{Z}_2$ *is* $\mathbf{Z}_2$*-equivalent to one of the following two:*

$$H \perp \dots \perp H \perp H \quad \text{or} \quad H \perp \dots \perp H \perp H' .$$

ii) *For any 2-adic unit* u *we have splittings over* $\mathbf{Z}_2$*:*

$$<u> \perp H \sim <u, 1, -1> ,$$
$$<u> \perp H' \sim <u-2, u+2, (2u+3)(u+2)^{-1}> ,$$
$$<2u> \perp H \sim <2u+8> \perp H' .$$

Proof of Theorem 1.2 for $p = 2$. By induction on k. Let $f(X) = 2^s v X^2$, $g(X) = 2^{s'} v' X^2$, $2 \nmid vv'$. By Proposition 1.1, $\theta(1, f, 2^t) = \theta(1, g, 2^t)$ for $t = s+2$, $s+3$ implies that $s = s'$ and $v \equiv v' \pmod 8$, hence $f \sim g$ over $\mathbf{Z}_2$. Let f and g be two forms in k variables, $k \geqslant 2$, of the same type. We consider the splittings over $\mathbf{Z}_2$:

$$f \sim 2^{s_1} f_1 \perp \dots \perp 2^{s_r} f_r ,$$
$$g \sim 2^{s_1} g_1 \perp \dots \perp 2^{s_r} g_r , \qquad 0 \leqslant s_1 < s_2 < \dots < s_r ,$$

f_i, g_i with unit determinant and the same number of variables, k_i, for all i. Without restriction we can suppose that f and g are primitive, that is $s_1 = 0$. If f and g have the same Gauss sums, then for each i, f_i and g_i are both properly or improperly primitive since, by Proposition 1.1, this is equivalent to the vanishing or not of $\theta(1, f, 2^{s_i+1})$. The proof proceeds in a different way according to whether f_1, g_1 are properly or improperly primitive.

Suppose that f_1 and g_1 are improperly primitive. If $k_1 > 2$, by i) of Lemma 1.3 we have, over $\mathbf{Z}_2$,

$$f \sim H \perp F , \qquad g \sim H \perp G ,$$

and, since $\theta(\ , H, 2^t)$ never vanishes, we have

$$\theta(\ , F, 2^t) = \frac{\theta(\ , f, 2^t)}{\theta(\ , H, 2^t)} = \theta(\ , G, 2^t) ,$$

for all t. By the induction hypothesis $F \sim G$, hence $f \sim g$ over $\mathbf{Z}_2$. If $k_1 = 2$ and $f_1 \sim g_1$ over $\mathbf{Z}_2$, we can proceed as above. Suppose that $k_1 = 2$ and

$$f \sim H \perp 2^{s_2} f_2 \perp \dots \perp 2^{s_r} f_r ,$$
$$g \sim H' \perp 2^{s_2} g_2 \perp \dots \perp 2^{s_r} g_r .$$

If $s_2 > 1$ (or $k = k_1 = 2$) or f_2, g_2 are improperly primitive we have

$$\theta(1, f, 4) = 2^{3+2(k-2)} = -\theta(1, g, 4) ,$$

a contradiction. Hence $s_2 = 1$ and f_2, g_2 are diagonalizable. By ii) of Lemma 1.3, $g \sim H \perp 2^{s_2} g'_2 \perp \dots$, over $\mathbf{Z}_2$, and we can proceed as above.

Suppose now that f_1 and g_1 are properly primitive. Let u be a 2-adic unit represented by f and g. We have splittings over $\mathbf{Z}_2$:

$$f \sim \langle u \rangle \perp F , \quad g \sim \langle u \rangle \perp G . \tag{1.2}$$

Since $\theta(\ , uX^2, 2^t) \neq 0$ for $t \neq 1$, we get $\theta(\ , F, 2^t) = \theta(\ , G, 2^t)$ for all $t \neq 1$. We have only to prove that $\theta(\ , F, 2) = \theta(\ , G, 2)$ and the claim will follow from the induction hypothesis. If $k_1 = 1$ or F and G are both properly or improperly primitive we are done. Assume that F is properly and G improperly primitive. This is possible indeed (see ii) of Lemma 1.3). By ii) of Lemma 1.3 we can always find a $\mathbf{Z}_2$-splitting $g \sim \langle u \rangle \perp G'$, with G' properly primitive except for the case that over $\mathbf{Z}_2$

$$g \sim \langle u \rangle \perp H' \perp 2^{s_2} g_2 \perp \dots ,$$

with $k_1 = 3$ and $s_2 > 1$ (or $k = 3$) or g_2 improperly primitive. Let us assume in this case that over $\mathbf{Z}_2$

$$f \sim \langle u, v, w \rangle \perp 2^{s_2} f_2 \perp \dots .$$

From $\theta(1, f, 4) = \theta(1, g, 4)$ we get

$$\exp(2\pi i (v + w)/8) = -\left(\frac{2}{vw}\right),$$

or, equivalently, $vw \equiv 3 \pmod 8$. This implies that either v or w are congruent (mod 8) to any of $u - 2, u + 2$; hence, changing u by v or w we get a splitting (1.2) with F and G both properly primitive. □

Remark. For $p = 2$ and $k \leqslant 4$ we could remove in the theorem the condition of f and g being of the same type. For $k \geqslant 5$ this is not possible anymore, as the following example shows: By Proposition 1.1 the diagonal forms $f = <1, 2, 2, 2, 4>$, $g = <1, 1, 2, 4, 4>$ have the same Gauss sums $\theta(\ , f, 2^t) = \theta(\ , g, 2^t)$ for all $t \geqslant 1$, however they are obviously not $\mathbf{Z}_2$-equivalent.

The theory of Minkowski reproduced in this section was extended by O'Meara to integral quadratic forms over local fields.

§ 2. Local representation masses and $\mathbf{Z}_p$-equivalence of forms

We identify $\mathbf{Q}_p$ with its topological dual by defining $<n, m> = \chi_p(nm)$, where χ_p is Tate's character:

$$\chi_p(a) = \exp(2\pi i \sum_{s<0} a_s p^s),$$

if $a = \sum_{s \geqslant s_o} a_s p^s$. Let dn be the Haar measure of $\mathbf{Q}_p$ normalized by $dn(\mathbf{Z}_p) = 1$. As is well-known, dn is selfdual. Let dx be the Haar measure of $\mathbf{Q}_p$ naturally induced by dn.

Let f be a non-singular integral p-adic quadratic form in $k \geqslant 1$ variables. We shall deal in this section with the representation mass function given by (0.1) for $\phi = 1_{(\mathbf{Z}p)^k}$. That is, we define for all $n_o \in \mathbf{Q}_p$:

$$r(n_o, f, \mathbf{Z}_p) = \lim_{U \to \{n_o\}} \left(dx(f^{-1}(U) \cap \mathbf{Z}_p^k)/dnU\right),$$

whenever this limit exists. Clearly r has support contained in $\mathbf{Z}_p$. We can also consider the Gauss-Weil transform of $1_{(\mathbf{Z}p)^k}$ by f given by

$$\theta(m, f, \mathbf{Q}_p) = \int_{\mathbf{Z}_p^k} <f(x), m> dx .$$

The relationship between these representation masses and the ones introduced in the preceeding section is given in the following

Lemma 2.1. i) *Let* $n \in \mathbf{Z}_p, n \neq 0$, *and* $t > v_p(4n)$. *Then*

$$r(n, f, \mathbf{Z}_p) = \lim_{s \to \infty} p^{(1-k)s} r(n, f, p^s) = p^{(1-k)t} r(n, f, p^t) .$$

ii) *Let* $m \in \mathbf{Z}_p$ *and* $u \in \mathbf{Z}_p, t \geqslant 1$ *be chosen arbitrarily satisfying* $m = up^{-t}$. *Then*

$$\theta(m, f, \mathbf{Q}_p) = p^{-kt}\theta(u, f, p^t) .$$

Proof. i) Let $U_t = n + p^t\mathbf{Z}_p$. We have $dn(U_t) = p^{-t}$ and

$$dx(f^{-1}(U_t)\cap\mathbf{Z}_p^k) = \sum_{a\in(\mathbf{Z}/p^t\mathbf{Z})^k} dx(f^{-1}(U_t)\cap(a+p^t\mathbf{Z}_p^k)) = p^{-kt}r(n, f, p^t),$$

since $f^{-1}(U_t)\cap(a+p^t\mathbf{Z}_p^k)$ is equal to $a + p^t\mathbf{Z}_p^k$ or vacuous, according to $f(a) \equiv n \pmod{p^t}$ or not. This proves the first equality in i).

We want now to show that $p^{(1-k)s}\, r_{p^s}(n) = p^{(1-k)(s-1)}\, r_{p^{s-1}}(n)$, for all $s > t$. We know that

$$r(n, f, p^s) = p^{-s}\sum_{u=1}^{p^s} \theta(u, f, p^s)\exp(-2\pi iunp^{-s}).$$

Let us denote by A and B the sum of the terms satisfying $p \mid u$ and $p \nmid u$, respectively. Clearly $A = p^{k-1}\, r(n, f, p^{s-1})$; hence, we are reduced to proving $B = 0$. Taking into account the explicit computations of Gauss sums (Proposition 1.1), we can express the sum B as

$$B = \begin{cases} C \displaystyle\sum_{u\in(\mathbf{Z}/p^s\mathbf{Z})^*} \left(\frac{u}{p}\right)^a \exp(-2\pi iunp^{-s}) & \text{if} \quad p > 2 \\ D \displaystyle\sum_{u\in(\mathbf{Z}/2^s\mathbf{Z})^*} \left(\frac{2}{u}\right)^b \exp\left(\frac{2\pi iu}{8}\right)^c \exp(-2\pi iun2^{-s}) & \text{if} \quad p = 2, \end{cases}$$

where C, D, a, b, c depend on f and s, but are independent of u. Now, $\exp(-2\pi inp^{-s})$ is a primitive p^l-th root of 1 with $l > 1$ if $p > 2$, and $l > 3$ if $p = 2$. One can check that, for any function φ defined on $(\mathbf{Z}/p^mZ)^*$, $m \geqslant 1$ and ξ any primitive p^l-th root of 1, $l > m$, one has

$$\sum_{u\in(Z/p^l\mathbf{Z})^*} \varphi(u)\xi^u = 0.$$

In particular, B must be zero.

In order to prove ii) we need only to observe that

$$\theta(m, f, \mathbf{Q}_p) = \int_{\mathbf{Z}_p^k} \exp(2\pi if(x)up^{-t})dx$$

$$= \sum_{a\in(\mathbf{Z}/p^t\mathbf{Z})^k} \exp(2\pi if(a)up^{-t}) \int_{a+p^t\mathbf{Z}_p^k} dx = p^{-kt}\theta(u, f, p^t). \qquad \square$$

Remark. After Siegel [13], it was very well known that for $n \neq 0$ the values $p^{(1-k)t}\, r(n, f, p^t)$ become constant for $t > 2v_p(4n)$. Lemma 2.1 shows that the minimum value of t with this property can be taken equal to half of the one found by Siegel.

By Lemma 2.1, $r(\ ,f,\mathbf{Z}_p)$ is locally constant, hence continuous on $\mathbf{Q}_p^*$, and $r(n,f,\mathbf{Z}_p)=0$ if and only if n is not represented by f in $\mathbf{Z}_p$. The fundamental fact is that r is integrable on $\mathbf{Z}_p$ and θ is its Fourier transform. This is well-known [4]. For the sake of completeness we give a short proof of this result using only the background introduced up to now.

PROPOSITION 2.2. *$r\in L^1(\mathbf{Z}_p)$ and*

$$\theta(m,f,\mathbf{Q}_p)=\int_{\mathbf{Z}_p} r(n,f,\mathbf{Z}_p)<n,m>dn\,.$$

Proof. We assume $p>2$. For $p=2$ the proof works in the same way with minor modifications left to the reader. Let $m=up^{-s}$, $u\in\mathbf{Z}_p$, $s\geqslant 0$. For all $t>s$, $\mathbf{Z}_p\backslash p^t\mathbf{Z}_p$ is compact, hence $r(n)$, being continuous, is integrable and we have by Lemma 2.1:

$$\int_{Z_p\backslash p^t\mathbf{Z}_p} r(n,f,\mathbf{Z}_p)<n,m>dn=\sum_{\substack{a\in Z/p^t\mathbf{Z}\\ a\neq 0}}\int_{a+p^t\mathbf{Z}_p} r(n,f,\mathbf{Z}_p)<n,m>dn$$

$$=\sum_{\substack{a\in Z/p^t\mathbf{Z}\\ a\neq 0}} p^{-kt}\,r(a,f,p^t)\exp(2\pi iaup^{-s})=p^{-kt}\big(\theta(p^{t-s}u,f,p^t)-r(0,f,p^t)\big)$$

$$=\theta(m,f,\mathbf{Q}_p)-p^{-kt}r(0,f,p^t)\,.$$

Both assertions of the proposition are consequences of Lebesgue's dominated convergence theorem if $p^{-kt}r(0,f,p^t)$ tends to zero as t tends to infinity. This is checked immediately for $k=1$. For $k>1$ it can be easily deduced from (1.1) and the explicit computation of Gauss sums in the preceding section. □

We are ready to prove a crucial fact for the rest of the paper:

THEOREM 2.3. *Let f,g be two non-singular integral p-adic quadratic forms in k variables. If $p=2$, assume that they are of the same type. The following conditions are equivalent:*

i) $f\sim g$ *over* $\mathbf{Z}_p$,

ii) $r(\ ,f,\mathbf{Z}_p)=r(\ ,g,\mathbf{Z}_p)$,

iii) $\theta(\ ,f,\mathbf{Q}_p)=\theta(\ ,g,\mathbf{Q}_p)$.

Proof. If $f\sim g$ over $\mathbf{Z}_p$, then $f\sim g$ over $\mathbf{Z}/p^t\mathbf{Z}$ and $r(\ ,f,p^t)=r(\ ,g,p^t)$ for all $t\geqslant 1$. By Lemma 2.1 this implies ii). By Proposition 2.2, ii) implies iii). Again by Lemma 2.1, iii) implies that $\theta(\ ,f,p^t)=\theta(\ ,g,p^t)$ for all $t\geqslant 1$, therefore condition i) follows now from Theorem 1.2. □

Let K be a local field and f a non-singular quadratic form in k variables defined over K. If ϕ is a Schwartz-Bruhat function on K^k, the representation mass function $r_\phi(\ , f, K)$ defined as in (0.1) coincides with another classical representation mass function introduced by Weil. This is Weil's procedure (see [4] for the details): for $n \neq 0$, the $(k-1)$-differential forms

$$\omega_i(x) = (-1)^{i-1} (D_i f)^{-1} \, dx_1 \wedge \dots \wedge \hat{dx_i} \wedge \dots \wedge dx_k ,$$

induce a gauge form ω_n on the affine variety $f^{-1}(n)$. Since we are in a local field, ω_n induces a positive measure $|\omega_n|$ on $f^{-1}(n)$ such that for every continuous function φ on K^k with compact support not containing zero we have

$$\text{(2.1)} \qquad \int_{K^k} \varphi(x) dx = \int_K \left(\int_{f^{-1}(n)} \varphi \, |\omega_n| \right) dn .$$

The representation mass of $n \in K^*$ by f with respect to ϕ is then defined as

$$F_\phi(n) = \int_{f^{-1}(n)} \phi \, |\omega_n| .$$

This function is continuous and after (2.1) it is easy to prove that F_ϕ is integrable and its Fourier transform coincides with the Gauss-Weil transform:

$$\int_{K^k} \phi(x) < f(x), m > dx = \int_K F_\phi(n) < n, m > dn .$$

Let now $n_o \in K^*$ and let U be any open neighbourhood of n_o. From (2.1) it is also easy to justify that:

$$\int_{f^{-1}(U)} \phi(x) dx = \int_U F_\phi(n) dn .$$

Since F_ϕ is continuous and K is locally compact, we have also:

$$F_\phi(n_o) = \lim_{U \to \{n_o\}} \left(\int_U F_\phi(n) dn \Big/ \int_U dn \right) = r_\phi(n_o) ,$$

thus $F_\phi = r_\phi$ on K^*.

§ 3. Local representation masses and $\mathbf{Q}_p$-equivalence of forms

There is a formula due to Minkowski (cf. [9]) for $\theta(u, f, p^t)$ if t is large enough, in which appear the well-known pair of invariants determining the $\mathbf{Q}_p$-equivalence class of f. As a consequence of this formula (cf. Proposition 3.2), we shall obtain a characterization of $\mathbf{Q}_p$-equivalence of forms through local representation masses.

Let $s \geqslant 1$ be an integer and $X_s = \{m \in \mathbf{Q}_p \mid -v_p(m) > s\}$. For any integral p-adic quadratic form f we define the functions:

$$r_s(\ , f, \mathbf{Z}_p) = p^{-\delta(f)/2}(r(\ , f, \mathbf{Z}_p) - p^{(1-k)s} r(\ , f, p^s)),$$
$$\theta_s(\ , f, \mathbf{Q}_p) = p^{-\delta(f)/2}\theta(\ , f, \mathbf{Q}_p) \cdot 1_{X_s},$$

where $\delta(f) = v_p(\det f)$.

The reader may check that the function defined on $\mathbf{Q}_p^k \backslash \{0\}$ by

$$\phi_s(x) = p^{-\delta(f)/2}\left(1 - p^{(1-k)s}\frac{r(f(x), f, p^s)}{r(f(x), f, \mathbf{Z}_p)}\right) \cdot 1_{(\mathbf{Z}_p)^k}(x)$$

is integrable over $\mathbf{Q}_p$ and that $r_s = r_{\phi_s}, \theta_s = \theta_{\phi_s}$, so that these functions follow the general pattern mentioned in the introduction. Note that ϕ_s is not a Schwartz-Bruhat function.

Proposition 3.1. *$r_s \in L^1(\mathbf{Z}_p)$ and* $\theta_s(m) = \int_{\mathbf{Z}_p} r_s(n) <m, n> dn$.

Proof. r_s is integrable since r and $r \pmod{p^s}$ are integrable. To prove the second assertion, by Proposition 2.2 we need only to compute

$$\hat{r}(m, f, p^s) = \int_{\mathbf{Z}_p} r(n, f, p^s) <m, n> dn\,.$$

Let $m = p^{-t}u$, $u \in \mathbf{Z}_p$, $p \nmid u$, $t \geqslant 0$. Let $t_o = \max\{s, t\}$. On each class $a + p^{t_o}\mathbf{Z}_p$, the integrand is constant and we have

$$\hat{r}(m, f, p^s) = p^{-t_o} \sum_{a \in \mathbf{Z}/p^{t_o}\mathbf{Z}} r(a, f, p^s) \exp(2\pi i u a p^{-t})\,.$$

If $t \leqslant s$ we have directly:

$$p^{(1-k)s}\,\hat{r}(m, f, p^s) = \theta(up^{s-t}, f, p^s) = \theta(m, f, \mathbf{Q}_p)\,.$$

If $t > s$ the sum is equal to

$$p^{-t} \sum_{a_o \in \mathbf{Z}/p^s\mathbf{Z}} r(a_o, f, p^s) \exp(2\pi i u a_o p^{-t}) \sum_{b \in \mathbf{Z}/p^{t-s}\mathbf{Z}} \exp(2\pi i u p^{s-t})^b = 0 . \qquad \square$$

In order to simplify Minkowski's formula for the theta-values, we will make use of the invariant $[f]_p$ of a p-adic quadratic form introduced by Conway [2]. Let α_k be the last invariant factor of f and let $s_o(f) = v_p(2p\alpha_k)$.

PROPOSITION 3.2. *Let f be a non singular p-adic integral quadratic form in k variables. For all $t \geqslant s_o(f)$ and $u \in \mathbf{Z}_p^*$ we have:*

$$\theta(u, f, p^t) = p^{(\delta+kt)/2} \varepsilon_p^{t^2(k+2\delta)} \left(\frac{u}{p}\right)^{kt+\delta} [f]_p \left(\frac{d_o}{p}\right)^t , \quad \text{if} \quad p \neq 2 ,$$

$$\theta(u, f, 2^t) = 2^{(\delta+k(t+1))/2} \exp(2\pi i k/8) [f]_2 \left(\frac{2}{d_o}\right)^t \left(\frac{2}{u}\right)^{kt} [u]_2^k (u, \det f)_2 ,$$
$$\text{if} \quad p = 2 .$$

Here $\delta = \delta(f)$, $d_o = p^{-\delta} \det f$ and $(a, b)_p$ denotes Hilbert's symbol.

Proof. Since $\theta(u, f, p^t) = \theta(1, uf, p^t)$, it is easy to reduce the claims to the case $u = 1$. Assume first $p > 2$. Let $v = v_1 \dots v_r$, $w = w_1 \dots w_{k-r}$, where

$$f \sim \perp_{1 \leqslant i \leqslant v} <p^{s_i} v_i> \perp_{1 \leqslant j \leqslant k-r} <p^{t_j} w_j>$$

over $\mathbf{Z}_p$, with s_i odd, t_j even, $v_i, w_j \in \mathbf{Z}_p^*$ for all i, j. Let $t > \max_{i,j} \{s_i, t_j\}$; by Prop. 1.1 we have

$$\theta(1, f, p^t) = p^{(\delta+kt)/2} \begin{cases} \varepsilon_p^r \left(\frac{v}{p}\right) & \text{if} \quad t \text{ even} \\ \varepsilon_p^{k-r} \left(\frac{w}{p}\right) & \text{if} \quad t \text{ odd} . \end{cases}$$

Since $[f]_p = \varepsilon_p^r \left(\frac{v}{p}\right)$, we get the desired formula.

We deal now with the case $p = 2$. Assume that, over $\mathbf{Z}_2$,

$$f \sim \perp_{1 \leqslant i \leqslant r} <2^{s_i} H_i> \perp_{1 \leqslant j \leqslant k-2r} <2^{t_j} v_j> ,$$

where H_i is 2-dimensional improperly primitive and $v_j \in \mathbf{Z}_2^*$. Let

$$U = \perp_{s_i \text{ even}} <H_i> , \; U' = \perp_{s_i \text{ odd}} <H_i> , \; V = \perp_{t_j \text{ even}} <v_j> , \; V' = \perp_{t_j \text{ odd}} <v_j> .$$

Let d, d', v, v' denote the respective determinants of U, U', V and V'. By Proposition 1.1 we have for all $t > 1 + \max_{i,j} \{s_i, t_j\}$

$$\theta(1, f, 2^t) = 2^{(\delta + k(t+1))/2} \exp(2\pi i w/8) \left(\frac{2}{dv}\right) \left(\frac{2}{d_o}\right)^t ,$$

where $w = \sum_{1 \leqslant j \leqslant k-2r} v_j$. Let s denote the dimension of U; one can see that

$$[U]_2 = \left(\frac{2}{d}\right)(-i)^{s/2} , \qquad [2U']_2 = (-i)^{(2r-s)/2} .$$

Let m be the number of v_j's in V congruent to 3 (mod 4), and let n_1, n_3, n_5, n_7 be the respective number of v_j's in V' congruent to 1, 3, 5 or 7 (mod 8); we have

$$[V]_2 \, [2V']_2 = i^{3n_1 + n_3 + 2n_5 + 3n_7} .$$

Summing up these expressions the result follows. □

Whereas $\mathbf{Z}_p$-equivalence of forms is determined by all functions $r(\ , f, p^t)$, $t \geqslant 1$ (Theorem 1.2), or equivalently by its limit value $r(\ , f, \mathbf{Z}_p)$ (Theorem 2.3), we prove in the next theorem that $\mathbf{Q}_p$-equivalent forms are characterized by having the same differences $r_s(\ , f, \mathbf{Z}_p)$ between these two functions, for s sufficiently large.

THEOREM 3.3. *Let f, g be non singular integral p-adic quadratic forms in k variables. Suppose that* $s \geqslant \max(s_o(f), s_o(g))$. *Then the following conditions are equivalent:*

i) $f \sim g$ *over* $\mathbf{Q}_p$,

ii) $r_s(\ , f, \mathbf{Z}_p) = r_s(\ , g, \mathbf{Z}_p)$,

iii) $\theta_s(\ , f, \mathbf{Q}_p) = \theta_s(\ , g, \mathbf{Q}_p)$.

Proof. For any integer $t \geqslant 1$ we consider the difference

$$\Delta r(n, f, p^t) := p^{(1-k)(t+1)} r(n, f, p^{t+1}) - p^{(1-k)t} r(n, f, p^t) .$$

It is clear from the definitions that

$$r_s(n, f, \mathbf{Z}_p) = p^{-\delta(f)/2} \sum_{t \geqslant s} \Delta r(n, f, p^t) .$$

If f and g are $\mathbf{Q}_p$-equivalent, then Proposition 3.2 implies that

$$p^{-\delta(f)/2} \theta(u, f, p^t) = p^{-\delta(g)/2} \theta(u, g, p^t) ,$$

for all $u \in \mathbf{Z}_p^*$, $t \geqslant s$. Let $n \in \mathbf{Z}_p$, since

$$\sum_{u \in (\mathbf{Z}/p^t\mathbf{Z})^*} p^{-t}\theta(u, f, p^t) \exp(-2\pi i n u p^{-t}) = r(n, f, p^t) - p^{k-1} r(n, f, p^{t-1})$$
$$= p^{(k-1)t} \Delta r(n, f, p^{t-1}),$$

we see at once that i) $\Rightarrow$ ii). By Proposition 3.1, ii) $\Rightarrow$ iii).

Assume now condition iii). Let $t = s, s+1$ and let $u \in \mathbf{Z}_p^*$; from the equality $\theta_s(up^{-t}, f, \mathbf{Q}_p) = \theta_s(up^{-t}, g, \mathbf{Q}_p)$ it follows, using Proposition 3.2, that $[f]_p = [g]_p$ and $\left(\frac{d_o(f)}{p}\right) = \left(\frac{d_o(g)}{p}\right)$. Since the forms f and g have the same discriminant and Conway invariant, they are equivalent over $\mathbf{Q}_p$. □

Next we devote a few lines to $\mathbf{R}$-equivalence. We identify $\mathbf{R}$ with its topological dual by defining $\langle n, m \rangle = \chi_\infty(n, m) := \exp(-2\pi i n m)$, for all $n, m \in \mathbf{R}$. We denote by dn, dx the Lebesgue measure on $\mathbf{R}$ and $\mathbf{R}^k$, respectively.

Let f be a non-singular real quadratic form in k variables with signature $(l, k-l)$. Let A be the matrix of f and let C be any matrix satisfying:

$$C^T A C = D, \quad D = \left(\begin{array}{c|c} I_l & 0 \\ \hline 0 & -I_{k-l} \end{array}\right).$$

$P := (CC^T)^{-1}$ is called a *majorant* of f. Since P is positive definite, the function

$$\phi_\infty(x) = |\det f|^{1/2} \exp(-\pi(x^T P x))$$

is a Schwartz function on $\mathbf{R}^k$. On $\mathbf{R}^*$ we define the functions

$$r(n, f, \mathbf{R}) = \lim_{U \to \{n\}} \left(\int_{f^{-1}(U)} \phi_\infty(x) dx / \text{vol } U \right),$$
$$\theta(m, f, \mathbf{R}) = \int_{\mathbf{R}^k} \phi_\infty(x) \langle f(x), m \rangle \, dx .$$

We have seen at the end of Section 2 that $r(\ , f, \mathbf{R})$ is a continuous function on $\mathbf{R}^*$, integrable on $\mathbf{R}$ and that $\theta(\ , f, \mathbf{R})$ is its Fourier transform. These functions do not depend on the chosen matrix C; they depend only on the signature of f. In fact, since $|\det C| = |\det f|^{-1/2}$, if we make the change of variables $x = Cy$ we obtain:

$$r(n, f, \mathbf{R}) = \lim_{U \to \{n\}} \left(\int_{d^{-1}(U)} \exp(-\pi(y^T y)) dy / dn(U) \right),$$

for all $n \in \mathbf{R}^*$, where we have denoted by d the quadratic form $d(x) = x^T Dx$. It is also easy to check that for all $m \in \mathbf{R}$ we have

$$\theta(m, f, \mathbf{R}) = \int_{\mathbf{R}^k} \exp\left(-\pi(y^T y) + 2\pi m i (y^T D y)\right) dy$$

$$= \left(\int_{\mathbf{R}} \exp\left(-\pi y^2(1+2im)\right) dy\right)^s \left(\int_{\mathbf{R}} \exp\left(-\pi y^2(1-2im)\right) dy\right)^{k-s}$$

$$= (1+2im)^{s/2}(1-2im)^{(k-s)/2}.$$

The following result is now clear:

THEOREM 3.4. *Let* f, g *be non-singular real quadratic forms in* k *variables. The following conditions are equivalent.*

i) $f \sim g$ *over* $\mathbf{R}$,

ii) $r(\ , f, \mathbf{R}) = r(\ , g, \mathbf{R})$,

iii) $\theta(\ , f, \mathbf{R}) = \theta(\ , g, \mathbf{R})$. □

§ 4. ADELIC REPRESENTATION MASSES

Let $\mathbf{A}$ be the ring of adeles over $\mathbf{Q}$. We identify $\mathbf{A}$ with its topological dual by defining $<n, m>$, where χ is Tate's character

$$\chi(a) = \chi_\infty(a_\infty) \cdot \prod_p \chi_p(a_p),$$

for any $a = (a_p) \in \mathbf{A}$. Let dn be the restricted product measure of the local measures used in the preceding sections. As is well-known, dn is also a selfdual measure. Let dx be the Haar measure on $\mathbf{A}^k$ naturally induced by dn.

A non-singular integral adelic quadratic form f in k variables with unit determinant can be identified to a collection (f_p) of non-singular integral p-adic quadratic forms in k variables such that $p \nmid \det f_p$, for almost all p.

Let Φ be the Schwartz-Bruhat function on $\mathbf{A}^k$ defined by

$$\Phi = \phi_\infty \cdot \prod_p 1_{(Z_p)^k}.$$

Let $\mathbf{A}_o := \mathbf{R} x \prod_p \mathbf{Z}_p$. We consider

$$\begin{aligned} r_\Phi(n, f, \mathbf{A}) := & \lim_{U \to (n)} \left(\int_{f^{-1}(U)} \Phi(x) dx \Big/ \int_U dn \right) \\ & = r(n_\infty, f_\infty, \mathbf{R}) \cdot \prod_p r(n_p, f_p, \mathbf{Z}_p), \end{aligned}$$

the limit being well-defined whenever the infinite product on the right is absolutely convergent. Applying Siegel's explicit formulas for $r(n_p, f_p, \mathbf{Z}_p)$ ([13, Hilfsatz 16]), it is easy to check that the product is absolutely convergent for all $n \in \mathbf{A}_o$ if $k \geqslant 5$. Since $r(\ , f_\infty, \mathbf{R}) \in L^1(\mathbf{R})$ and $\prod_p \mathbf{Z}_p$ is compact, r_Φ is an everywhere defined continuous function on $\mathbf{A}$, with support contained in $\mathbf{A}_o$, and integrable on $\mathbf{A}$. On the other hand, clearly $\Phi \in L^1(\mathbf{A}^k)$ and we have

$$\theta_\Phi(m, f, \mathbf{A}) := \int_{\mathbf{A}^k} \Phi(x) < f(x), m > dx = \theta(m_\infty, f_\infty, \mathbf{R}) \prod \theta(m_p, f_p, \mathbf{Q}_p).$$

Note that the infinite product is always well-defined since only a finite number of factors are different from 1.

We recall that given two integral adelic quadratic forms in k variables it is said that they belong to the same *genus* if $f_\infty \sim g_\infty$ over $\mathbf{R}$ and $f_p \sim g_p$ over $\mathbf{Z}_p$ for all p. We say that they are of the same 2-type if f_2 and g_2 are of the same type over $\mathbf{Z}_2$.

THEOREM 4.1. *Let f, g be two non-singular integral adelic quadratic forms in $k \geqslant 5$ variables with unit determinant. Assume that they are of the same 2-type. Then the following conditions are equivalent:*

i) $\operatorname{gen} f = \operatorname{gen} g$,

ii) $r_\Phi(\ , f, \mathbf{A}) = r_\Phi(\ , g, \mathbf{A})$,

iii) $\theta_\Phi(\ , f, \mathbf{A}) = \theta_\Phi(\ , g, \mathbf{A})$.

Proof. Two forms in the same genus have the same local integral representation masses, hence i) $\Rightarrow$ ii). Since θ_Φ is just the Fourier transform of r_Φ, ii) $\Rightarrow$ iii). Now condition iii) is equivalent to $\theta(\ , f_\infty, \mathbf{R}) = \theta(\ , g_\infty, \mathbf{R})$ and $\theta(\ , f_p, \mathbf{Q}_p) = \theta(\ , g_p, \mathbf{Q}_p)$ for all p; therefore by Theorems 2.3 and 3.4, iii) $\Rightarrow$ i). □

We deal now with $\mathbf{A}$-equivalence of forms. If f and g are two non-singular quadratic forms defined over $\mathbf{Q}$, we have by the Minkowski-Hasse theorem that $f \sim g$ over $\mathbf{Q}$ if and only if $f \sim g$ over $\mathbf{A}$. Thus Theorem 4.2 below can be also considered as a characterization of $\mathbf{Q}$-equivalence in terms of representation masses.

For every finite set S of primes and for every integer $s \geqslant 1$ we consider the following function defined on $\mathbf{A}$:

$$r_{S,s}(n, f, \mathbf{A}) = r(n_\infty, f_\infty, \mathbf{R}) \cdot \prod_{p \in S} r_s(n_p, f_p, \mathbf{Z}_p) \cdot \prod_{p \notin S} r(n_p, f_p, \mathbf{Z}_p).$$

As before, $r_{S,s}$ is well-defined, continuous and integrable if $k \geqslant 5$. The corresponding function $\theta_{S,s}(\ , f, \mathbf{A})$ will be well-defined and continuous for all $k \geqslant 1$, being the Fourier transform of the former.

Since $f \sim g$ over $\mathbf{A}$ is equivalent to $f_p \sim g_p$ over $\mathbf{Q}_p$ for all p including $p = \infty$, and $f_p \sim g_p$ over $\mathbf{Z}_p$ for almost all p, we get from Theorem 2.3, 3.3 and 3.4 the following:

THEOREM 4.2. *Let f, g be two non-singular integral adelic quadratic forms in $k \geqslant 5$ variables. Let $S = \{p\,; p \mid \det f_p \,.\, \det g_p\}$ and let $s \geqslant \max(s_o(f), s_o(g))$. Then the following conditions are equivalent:*

i) $f \sim g$ *over* $\mathbf{A}$,

ii) $r_{S,s}(\ , f, \mathbf{A}) = r_{S,s}(\ , g, \mathbf{A})$,

iii) $\theta_{S,s}(\ , f, \mathbf{A}) = \theta_{S,s}(\ , g, \mathbf{A})$. □

Note that we could have also expressed these functions as $r_{S,s} = r_{\Phi_{S,s}}$, $\theta_{S,s} = \theta_{\Phi_{S,s}}$, where $\Phi_{S,s} \in L^1(\mathbf{A}^k)$ is defined as:

$$\Phi_{S,s} = \phi_\infty \cdot \prod_{p \in S} \phi_s \cdot \prod_{p \notin S} 1_{(\mathbf{Z}_p)^k}.$$

§ 5. REPRESENTATION MASSES IN $\mathbf{Z}$

Let (V, q) be a regular quadratic space over $\mathbf{Q}$ of dimension k, and let G be the proper orthogonal group of this space. The adele group $G(\mathbf{A})$ operates in the set of lattices L of V; by definition the orbit of L under this action is called the genus of L. The orbit of L under the subgroup $G(\mathbf{Q})$ of $G(\mathbf{A})$ is the class of L.

If $L = \mathbf{Z}e_1 \oplus \ldots \oplus \mathbf{Z}e_k$ is a lattice of V, the formula

$$f(x_1, \ldots, x_k) = q(x_1 e_1 + \ldots + x_k e_k)$$

stablishes a one to one correspondence between the set of classes of lattices of (V, q) and the set of classes, over $\mathbf{Z}$, of quadratic forms which are $\mathbf{Q}$-equivalent to q.

For any $n \in \mathbf{Q}^*$, a representation of n by L is a couple (x, L) such that $x \in L$ and $q(x) = n$. Since the groups $G(\mathbf{Q})$, $G(\mathbf{A})$ operate on the set of such representations, one can group them in classes and genera, respectively.

For each $x \in q^{-1}(n)$, the stabilizer of G at x can be identified with the orthogonal group of the quadratic form induced by q on $<x>^{\perp}$. By Witt's theorem, the action of $G(\mathbf{Q})$ on $q^{-1}(n)$ is transitive. Suppose that $q^{-1}(n) \cap L \neq \emptyset$ and fix $x_o \in q^{-1}(n) \cap L$. Let us choose gauge forms ω on G and ω_{x_o} on g_{x_o}. If $\sigma \in G(\mathbf{Q})$ and $x = \sigma x_o$, we consider on $g_x = \sigma g_{x_o} \sigma^{-1}$ the gauge forms ω_x obtained from ω_{x_o} by pull back. Let μ_∞, μ_p, μ; $\mu_{x,\infty}, \mu_{x,p}, \mu_x$ be the respective local measures and Tamagawa measure induced by these gauge forms on G and g_x. The homogeneous space G/g_x can be identified with $q^{-1}(n)$ and there exists a unique gauge form ω' on $q^{-1}(n)$ such that if $\mu'_\infty, \mu'_p, \mu'$ denote the local measures and Tamagawa measure induced by ω', then $\mu = \mu_x \,.\, \mu'$ (cf. [18]).

The representation mass of n by (x, L) is defined in [6] as:

$$r(n, (x, L)) = \mu_{x,\infty}\big(g_x(\mathbf{R})/g_x(\mathbf{Q}) \cap G_L\big),$$

where G_L is the stabilizer of the lattice L in $G(\mathbf{Q})$. By the above normalization of gauge forms, this definition depends only of the class of (x, L). Thus, one can define the representation mass of n by L as

$$r(n, L) = \sum_x r(n, (x, L)),$$

x running over a system of representatives of the classes (x, L) with fixed L.

Let $L_1, \ldots, L_h$ be a system of representatives of the classes in the genus of L and let $G_i = G_{L_i}$. The mass of the genus of L is defined as

$$m(\text{gen } L) = \sum_{i=1}^{h} \mu_\infty\big(G(\mathbf{R})/G_i\big),$$

and the representation mass of n by the genus of L as

$$r(n, \text{gen } L) = m(\text{gen } L)^{-1} \sum_{i=1}^{h} r(n, L_i).$$

If q is definite and $k \geqslant 3$ or if q is indefinite and $k \geqslant 4$, the Tamagawa number of G and g_x is 2. From this fact it can be deduced (cf. [17]) that

$$r(n, \text{gen } L) = \prod_p \mu'_p\big(q^{-1}(n) \cap L_p\big),$$

where L_p is the localization of L at p.

From now on we shall assume that f is a **Z**-integral quadratic form *positive definite in* $k \geqslant 3$ *variables or indefinite in* $k \geqslant 4$ *variables.*

Let $L = \mathbf{Z}^k$. By normalizing ω or ω_{x_o}, we can assume that

$$\mu'_p(f^{-1}(n) \cap \mathbf{Z}_p^k) = r(n, f, \mathbf{Z}_p) .$$

Therefore, we obtain Siegel's formula:

$$r(n, \text{gen}\, \mathbf{Z}^k) = \prod_p r(n, f, \mathbf{Z}_p) . \tag{4.1}$$

The number on the left of (4.1) admits a quite natural interpretation in the definite case, due to the fact that the set $f^{-1}(n) \cap \mathbf{Z}^k$, the group $G_{\mathbf{Z}^k}$, and the three volumes which appear in the formula

$$\mu_\infty(G(\mathbf{R})) = \mu_{x,\infty}(g_x(\mathbf{R})) \,.\, \mu'_\infty(f^{-1}(n))$$

are all finite. In fact, defining $r(n, f) = \#(f^{-1}(n) \cap \mathbf{Z}^k)$, $o(f) = \# G_{\mathbf{Z}^k}$, and denoting by $f_1, \ldots, f_h$ a complete system of representatives of the classes of forms in the genus of f, from the above considerations it is not hard to deduce the following set of formulas in the definite case:

$$r(n, \mathbf{Z}^k) = \frac{\mu_\infty(G(\mathbf{R}))}{\mu'_\infty(f^{-1}(n))} \cdot \frac{r(n, f)}{o(f)} ,$$

$$m(\text{gen}\, \mathbf{Z}^k) = \mu_\infty(G(\mathbf{R})) \sum_{i=1}^{h} o(f_i)^{-1} ,$$

$$r(n, \text{gen}\, \mathbf{Z}^k) = \mu'_\infty(f^{-1}(n))^{-1} \Big(\sum_{i=1}^{h} r(n, f_i) o(f_i)^{-1}\Big) \Big/ \Big(\sum_{i=1}^{h} o(f_i)^{-1}\Big) .$$

Moreover, the factor $\mu'_\infty(f^{-1}(n))$ is, by definition, equal to the function $F_\Phi(n, f, \mathbf{R})$ for $\Phi = 1$ (see the end of Section 2). Hence, we have

$$\mu'_\infty(f^{-1}(n)) = \lim_{U \to \{n\}} \big(\text{vol}(f^{-1}(U))/\text{vol}\, U\big) .$$

We recover in this way Siegel's real density of representations [13], which has the well-known value:

$$\mu'_\infty(f^{-1}(n)) = \pi^{k/2} \Gamma(k/2)^{-1} (\det f)^{-1/2} n^{k/2-1}$$

if $n > 0$ (if $n < 0$ is $\mu'_\infty(f^{-1}(n)) = 0$).

In order to be coherent with the classical notation, we define the integral representation masses $r(n, f)$, $r(n, \text{gen}\, f)$ in a different way, according f to be definite or indefinite.

$$r(n, f) := \begin{cases} \#(f^{-1}(n) \cap \mathbf{Z}^k) & \text{if} \quad f \text{ definite} \\ r(n, \mathbf{Z}^k) & \text{if} \quad f \text{ indefinite} \end{cases}$$

$$r(n, \text{gen } f) := \begin{cases} r(n, \text{gen } \mathbf{Z}^k) \,.\, \mu'_\infty(f^{-1}(n)) & \text{if} \quad f \text{ definite} \\ r(n, \text{gen } \mathbf{Z}^k) & \text{if} \quad f \text{ indefinite}\,. \end{cases}$$

Let us denote, moreover, $\mu(f) = \mu_\infty(G(\mathbf{R})/G_{\mathbf{Z}^k})$.

It is a well-known fact that, in the indefinite case, and for all $n \in \mathbf{Z}\backslash\{0\}$

$$r(n, \text{gen } f) = \mu(f)^{-1} r(n, f)\,,$$

since the average representation mass in a spinor genus coincides with $r(n, \text{gen } f)$, but for $k \geqslant 4$ there is only one class in each spinor genus.

Summing up all this considerations we can rewrite Siegel's formula in the form:

$$r(n, \text{gen } f) = \mu'_\infty(f^{-1}(n)) \,.\, \prod_p r(n, f, \mathbf{Z}_p)$$

if f is definite,

$$r(n, f) = \mu(f) \,.\, \prod_p r(n, f, \mathbf{Z}_p)\,,$$

otherwise.

We can now reproduce partially the outline of the preceding sections. Considering $r(\ , f)$, $r(\ , \text{gen } f)$ as functions defined on $\mathbf{Z}$, we can define theta series by taking the formal Fourier transform:

$$\theta(z, f) = \sum_{n \geqslant 0} r(n, f) \exp(\pi i n z)\,,$$

$$\theta(z, \text{gen } f) = \sum_{n \geqslant 0} r(n, \text{gen } f) \exp(\pi i n z)\,;$$

and zeta functions by taking formal Mellin transforms:

$$\zeta(s, f) = \sum_{n > 0} r(n, f) n^{-s}\,,$$

$$\zeta(s, \text{gen } f) = \sum_{n > 0} r(n, \text{gen } f) n^{-s}\,.$$

Both functions have been largely investigated. We recall next their more relevant properties for our purposes (cf. [13], [14], [15], [12], [10]). If f is definite, $\theta(z, f)$ is a modular form of weight $k/2$, with character, with respect to a congruence group $\Gamma_o(N)$. It satisfies the functional equation

(5.1) $$\theta(z, f) = (\det f)^{-1/2}(-iz)^{-k/2}\theta(-1/z, f^\sharp)\,,$$

where $f^\sharp$ denotes the quadratic form associated to the dual lattice of $\mathbf{Z}^k$

in $\mathbf{R}^k$ with respect to f. And $\theta(z, \text{gen } f)$ is an Eisenstein series for the same group.

The Dirichlet series defining $\zeta(s, f)$ converges, both in the definite and in the indefinite case, for $\operatorname{Re} s > k/2$. It has a meromorphic continuation to the whole plane with a simple pole at $s = k/2$ (and possibly at $s = 1$, if f is indefinite) and it satisfies a functional equation involving $\zeta(s, f)$ and $\zeta(k/2 - s, f^{-1})$. Clearly the zeta function $\zeta(\ , \text{gen } f)$ has the same properties.

In the indefinite case, the residue at $s = k/2$ of these zeta functions is given by:

$$[\zeta(s, f)]_{k/2} = 2\rho_k \,|\det f\,|^{k/2} \mu(f)\,, \tag{5.2}$$

$$[\zeta(s, \text{gen } f)]_{k/2} = 2\rho_k \,|\det f\,|^{k/2}\,, \tag{5.3}$$

where

$$\rho_k := \sum_{j=1}^{k-1} \Gamma(j/2)\pi^{-j/2}\,.$$

THEOREM 5.1. *Let f, g be two non singular* **Z**-*integral quadratic forms. Suppose that $f \sim g$ over* **R** *and that they are of the same* 2-*type if $k \geqslant 5$. Then the following conditions are equivalent:*

i) $\text{gen } f = \text{gen } g$,

ii) $r(\ , \text{gen } f) = r(\ , \text{gen } g)$,

iii) $\zeta(\ , \text{gen } f) = \zeta(\ , \text{gen } g)$.

Proof. It is clear from the definitions that i) $\Rightarrow$ ii) $\Rightarrow$ iii). Assume that iii) is satisfied and let us show first that it must be $\det f = \det g$. In the indefinite case, this is a direct consequence of (5.3) and the fact that $f \sim g$ over **R**. In the definite case, and since iii) is equivalent to the equality $\theta(\ , \text{gen } f) = \theta(\ , \text{gen } g)$, by (5.1) we have

$$(\det f)^{-1/2}\, \theta(\ , \text{gen } f^{\#}) = (\det g)^{-1/2}\, \theta(\ , \text{gen } g^{\#})\,.$$

Since $f^{\#}, g^{\#}$ are two definite quadratic forms we have

$$\lim_{t \to \infty} \theta(it, \text{gen } f^{\#}) = \lim_{t \to \infty} \theta(it, \text{gen } g^{\#}) = 1\,,$$

hence $\det f = \det g$ and, moreover, $\mu'_\infty(f^{-1}(n)) = \mu'_\infty(g^{-1}(n))$.

By Siegel's formula, we see that condition iii) implies, in both cases, that

$$\prod_p r(n, f, \mathbf{Z}_p) = \prod_p r(n, g, \mathbf{Z}_p)$$

for all $n \neq 0$. Let now S be any finite set of primes including those p dividing $2 \det f$. Assume $n \in \mathbf{Z}\backslash\{0\}$. If $p \notin S$ we have by [13, Hilfsatz 16] that

$$r(n, f, \mathbf{Z}_p) = r(n, g, \mathbf{Z}_p) \neq 0 .$$

Therefore, by [13, Hilfsatz 25] we have:

$$\prod_{p \in S} r(n, f, \mathbf{Z}_p) = \prod_{p \in S} r(n, g, \mathbf{Z}_p) .$$

Let $\mathbf{Z}_S = \prod_{p \in S} \mathbf{Z}_p$. By the chinese remainder theorem we get the equality of functions over $\mathbf{Z}_S$:

$$\prod_{p \in S} r(\ , f, \mathbf{Z}_p) = \prod_{p \in S} r(\ , g, \mathbf{Z}_p) .$$

Since

$$\begin{aligned}\prod_{p \in S} \theta(m_p, f, \mathbf{Z}_p) &= \prod_{p \in S} \int_{\mathbf{Z}_p} r(n_p, f, \mathbf{Z}_p) < n_p, m_p > dn_p \\ &= \int_{\mathbf{Z}_S} \Big(\prod_{p \in S} r(n_p, f, \mathbf{Z}_p) < n_p, m_p > \Big) \Big(\bigotimes_{p \in S} dn_p\Big) \\ &= \int_{\mathbf{Z}_S} \Big(\prod_{p \in S} r(n_p, f, \mathbf{Z}_p)\Big) < n_S, m_S > dn_S ,\end{aligned}$$

where n_S, m_S, dn_S have their natural meanings, we see that condition iii) implies

$$\prod_{p \in S} \theta(\ , f, \mathbf{Q}_p) = \prod_{p \in S} \theta(\ , g, \mathbf{Q}_p) .$$

Taking into account that $\theta(\mathbf{Z}_p, f, \mathbf{Q}_p) = 1$, we get that $\theta(\ , f, \mathbf{Q}_p) = \theta(\ , g, \mathbf{Q}_p)$, for all $p \in S$. Thus, by applying Theorem 2.3 we get that $f \sim g$ over $\mathbf{Z}_p$, for all p. □

We have proved that the representation mass function $r(\ , \operatorname{gen} f)$ determines the genus of f under certain conditions on the ∞-type and the 2-type of f. The following examples of forms f, g such that $r(\ , \operatorname{gen} f) = r(\ , \operatorname{gen} g)$ but not belonging to the same genus, show that none of these conditions can be dropped (cf. also [5]).

Examples. We consider $I = \perp_{1 \leqslant i \leqslant 4} <1>$, $J = \perp_{1 \leqslant i \leqslant 4} <-1>$. Let $f = I \perp I \perp J$ and $g = I \perp J \perp J$. These two forms satisfy $f \sim g$ over $\mathbf{Z}_p$ for all p, but they are not $\mathbf{R}$-equivalent. Let $f = <1, 1, 2, 4, 4>$,

$g = <1, 2, 2, 2, 4>$ or $f = <-1, 1, 2, 4, 4>$ and $g = <-1, 2, 2, 2, 4>$. In both cases f and g are **R**-equivalent and satisfy $r(\ ,\mathrm{gen}\, f) = r(\ ,\mathrm{gen}\, g)$, but they are not $\mathbf{Z}_2$-equivalent.

In the following theorem we show that, both in the definite and in the indefinite case, two quadratic forms with the same representation numbers must belong to the same genus. In the low dimensional cases ($k=2$ or $k=3$, indefinite) an analogous result can also be stated. If $k = 3$ and the forms are indefinite, the proof requires a finer study of their representation masses (cf. [11]). If $k = 2$ much more is true, since two **Z**-integral quadratic forms with the same 2-type which represent the same set of integers belong already to the same genus.

THEOREM 5.2. *Let f, g be two non-singular* **Z***-integral quadratic forms in k variables. Suppose that $f \sim g$ over* **R** *and that f and g are of the same 2-type if $k \geqslant 5$. Then $r(\ , f) = r(\ , g)$ implies that f and g belong to the same genus.*

Proof. $k \geqslant 3$, *f definite.* By hypothesis we have $\theta(\ , f) = \theta(\ , g)$ as functions on the upper half-plane. As is well-known, $\theta(\ , f) - \theta(\ ,\mathrm{gen}\, f)$ is a cusp form. Thus $\theta(\ ,\mathrm{gen}\, f) - \theta(\ ,\mathrm{gen}\, g)$, being both a cusp form and an Eisenstein series, must be zero. Applying Theorem 5.1 we have that $\mathrm{gen}\, f = \mathrm{gen}\, g$.

$k \geqslant 4$, *f indefinite.* Since $r(\ ,\mathrm{gen}\, f) = \mu(f)^{-1} r(\ , f)$, we need only to show that $\mu(f) = \mu(g)$ and apply Theorem 5.1. By hypothesis $\zeta(\ , f) = \zeta(\ , g)$; from the residue formula (5.2) we get

$$|\det f|^{-1/2}\, \mu(f) = |\det g|^{-1/2}\, \mu(g)\,. \tag{5.4}$$

There is an explicit relation between $\mu(f)$ and the volume $V(f)$ of the majorante space [16, p. 110] which, together with the fact $V(f) = V(f^{-1})$ furnishes the relation

$$\mu(f^{-1}) = |\det f|^{k+1}\, \mu(f)\,.$$

Now, from the functional equation of the zeta function [14], it is easily deduced that $\mu(f^{-1}) = \mu(g^{-1})$, hence

$$|\det f|^{k+1}\, \mu(f) = |\det g|^{k+1}\, \mu(g)\,.$$

This together with (5.4) implies $|\det f| = |\det g|$ and $\mu(f) = \mu(g)$. □

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