3. Smooth singular homology

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and second, that the map a_X introduced in (2.4) induces an isomorphism

$$a : \Gamma_c(X, \Omega^{\cdot\vee}) \xrightarrow{\sim} D^c_{\cdot}(X, \mathbf{C}) . \tag{2.7}$$

Collect this together to conclude the proof. Q.E.D.

The de Rham homology as defined here agrees with the original theory based on currents [6]: the inclusion of the complex of currents in $\Omega.^{\vee}$ is a quasi-isomorphism as can be seen by the method used in the last third of the proof above.

As a consequence of the isomorphism (2.7) we can of course redefine de Rham homology as

$$H^c_p(X, \mathbf{C}) = H_p\Gamma_c(X, \Omega^{\cdot\vee}) . \tag{2.8}$$

If the letter c is dropped we obtain Borel Moore homology, compare [5] IX and the references given there.

The biduality theorem 2.1 is certainly related to that of Verdier [7], [1]. In fact most of the material presented here may be extended to a context of similar generality. I hope to return to this point in the near future.

3. Smooth singular homology

Let us consider an n-dimensional smooth manifold X. Integration over smooth singular simplexes defines a map

$$S^{\infty}_{\cdot}(X, \mathbf{C}) \to D^c_{\cdot}(X, \mathbf{C}) \tag{3.1}$$

from the complex of smooth singular simplexes to the complex of compact chains on X.

(3.2) Theorem. *Integration induces an isomorphism*

$$H^{\infty}_{\cdot}(X, \mathbf{C}) \xrightarrow{\sim} H^c_{\cdot}(X, \mathbf{C})$$

from smooth singular homology to de Rham homology.

Proof. Let us first discuss *Mayer-Vietoris* sequences in de Rham homology. For open subsets U and V of X a Mayer-Vietoris sequence arises from the following exact sequence of complexes

$$0 \to \Gamma_c(U \cap V, \Omega^{\cdot\vee}) \xrightarrow{+} \Gamma_c(U, \Omega^{\cdot\vee}) \oplus \Gamma_c(V, \Omega^{\cdot\vee}) \xrightarrow{-} \Gamma_c(U \cup V, \Omega^{\cdot\vee}) \to 0 . \tag{3.3}$$

The main reason for the exactness of the sequence is

$$H_c^1(U \cap V, \Omega^{q\vee}) = 0\,, \quad q \in \mathbf{N}\,,$$

compare [5] III. 7.5. The vanishing of the cohomology group follows from the fact that $\Omega^{q\vee}$ is a flabby sheaf.

In singular homology the Mayer-Vietoris sequence originates from the tautological exact sequence

$$0 \to S_{\bullet}^{\infty}(U \cap V, \mathbf{C}) \to S_{\bullet}^{\infty}(U, \mathbf{C}) \oplus S_{\bullet}^{\infty}(V, \mathbf{C}) \to S_{\bullet}^{\infty}(U, V; \mathbf{C}) \to 0\,,$$

where $S_{\bullet}^{\infty}(U, V; \mathbf{C})$ is the complex based on singular simplexes supported entirely by U or entirely by V. The difficult part is to prove that the inclusion

$$S_{\bullet}^{\infty}(U, V, \mathbf{C}) \to S_{\bullet}^{\infty}(U \cup V, \mathbf{C})$$

is a quasi-isomorphism, compare [8].

This description makes it obvious that the Mayer-Vietoris sequences in the two theories are connected by a commutative ladder.

A second common feature of the two theories is that given a manifold X which is the disjoint union of the family (X_α) of open subsets, then

$$\bigoplus_{\alpha} H.(X_\alpha, \mathbf{C}) \xrightarrow{\sim} H.(X, \mathbf{C}) \tag{3.4}$$

as it follows from the Borel-Heine theorem.

Let us now investigate the case $X = \mathbf{R}^n$. The homology groups are for both theories, compare (2.1)

$$H_0(\mathbf{R}^n, \mathbf{C}) = \mathbf{C}\,, \quad H_i(\mathbf{R}^n, \mathbf{C}) = 0\,, \quad i \geqslant 1\,.$$

The transition from $H_{\bullet}^{\infty}(\mathbf{R}^n, \mathbf{C})$ to $H_{\bullet}^{c}(\mathbf{R}^n, \mathbf{C})$ is an isomorphism as it follows from the discussion of zero cycles at the end of section 1, see also (2.1).

The result follows by *Fribourg-induction*, an elementary but ingenious induction procedure based on the Mayer-Vietoris sequence, see [2] or [4] where this method is used for the proof of Poincaré duality and the Künneth theorem. The last reference attributes this method to the master thesis of C. Auderset, Fribourg 1968. Q.E.D.

Let us record that the canonical isomorphism from smooth singular homology to singular homology

$$H_{\bullet}^{\infty}(X, \mathbf{C}) \xrightarrow{\sim} H.(X, \mathbf{C}) \tag{3.5}$$

likewise can be established by Fribourg-induction.

If we combine theorem (3.2) and the biduality theorem (2.1) we obtain what is usually known as the

(3.6) DE RHAM THEOREM. *Integration over smooth singular simplexes induces an isomorphism*

$$H^{\bullet}(X, \mathbf{C}) \xrightarrow{\sim} H^{\bullet}_{\infty}(X, \mathbf{C})$$

from de Rham cohomology to smooth singular cohomology.

4. RELATIVE DE RHAM HOMOLOGY

Let us start by some general remarks on the support of a compact p-chain T on a smooth n-dimensional manifold X. Since we can realize T as a section in the sheaf $\Omega^{p\vee}$ the general sheaf theoretic notion of support applies: The *support* of T, $\operatorname{Supp}(T)$ is the smallest closed subset Z of X, such that the restriction of T to $X - Z$ is zero.

(4.1) EXAMPLE. Integration over an oriented compact p-dimensional submanifold K with boundary defines a compact p-chain κ with $\operatorname{Supp}(\kappa) = K$. From Stokes formula

$$\int_K d\omega = \int_{\partial K} \omega\,, \quad \omega \in \Gamma(X, \Omega^p)\,,$$

we conclude that $\operatorname{Supp}(b\kappa) = \partial K$.

Let us now consider the inclusion $j: U \to X$ of an open subset U of X. The induced map

$$j_*: D_p^c(U, \mathbf{C}) \to D_p^c(X, \mathbf{C})\,, \quad p \in \mathbf{N}\,,$$

is injective since we may interpret j_* as "extension by zero" in the sheaf $\Omega_p^{\vee}$, compare (2.5). A compact p-chain T on X belongs to the image of j_* if and only if $\operatorname{Supp}(T) \subseteq U$. The complex $D_{\bullet}^c(X, U; \mathbf{C})$ of *relative compact chains* is defined to fit into the following exact sequence

(4.2) $$0 \to D_{\bullet}^c(U, \mathbf{C}) \xrightarrow{j_*} D_{\bullet}^c(X, \mathbf{C}) \to D_{\bullet}^c(X, U; \mathbf{C}) \to 0\,.$$

On this basis we can define the *relative de Rham* homology group

$$H_p(X, U; \mathbf{C}) = H_p D_{\bullet}^c(X, U; \mathbf{C})\,, \quad p \in \mathbf{N}\,.$$