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If $\phi \in H^\infty(\Omega)$ is valued in $\mathbf{R}^{n(n+1)/2}$, let us consider it as a function valued in $\mathbf{R}^{n(n+3)/2}$ by adding n components $\phi_j = 0$ for $1 \leq j \leq n$, and define $\psi(u)\phi$ as a continuous extension to \mathbf{R}^n of the function

$$(16) \quad v = -\frac{1}{2} A(u)^{-1} \phi$$

where $A(u)$ is the $n(n+3)/2$ square matrix the rows of which are $\partial_j u$ for $1 \leq j \leq n$ and $\partial_j \partial_k u$ for $1 \leq j \leq k \leq n$; thanks to our choice of u_0 , the matrix $A(u_0)$ is invertible on Ω , and so is $A(u)$ for any u close enough to u_0 . Since $A(u)^{-1}$ is an algebraic function of derivatives of u up to order 2, estimates such as (3) are again classical.

Finally, we have to prove that this operator ψ inverts ϕ' (formula (2)). Applying $A(u)$ to the function v in (16), one gets

$$\begin{aligned} \langle \partial_j u, v \rangle &= -\frac{1}{2} \phi_j = 0 & 1 \leq j \leq n \\ \langle \partial_j \partial_k u, v \rangle &= -\frac{1}{2} \phi_{jk} & 1 \leq j \leq k \leq n. \end{aligned}$$

The x_k derivative of the first equation gives $\langle \partial_j \partial_k u, v \rangle + \langle \partial_j u, \partial_k v \rangle = 0$, and one gets also $\langle \partial_j \partial_k u, v \rangle + \langle \partial_k u, \partial_j v \rangle = 0$ so that the second equation and (15) give $\phi'(u)v = \phi$ in Ω .

Thus all the assumptions of the theorem are fulfilled, and it follows that we can get a solution if $\phi(u_0)$ is sufficiently small in some $H^s(\Omega)$ norm; but according to (14), $\phi(u_0) = g^0 - g$, and the result is that (13) can be solved for any metric g close enough to g^0 , as required.

APPENDIX:

CONSTRUCTION OF THE SMOOTHING OPERATORS IN SOBOLEV SPACES

Let us recall that $v \in H^s(\mathbf{R}^n)$ means $v \in \mathcal{S}'(\mathbf{R}^n)$ and

$$|v|_s^2 = (2\pi)^{-n} \int (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 d\xi < \infty.$$

Let $\chi: \mathbf{R}^n \rightarrow [0, 1]$ be a C^∞ function taking the value 1 in a neighborhood of 0 and vanishing for $|\xi| \geq \sqrt{3}$. For $v \in H^\infty(\mathbf{R}^n)$ and $\theta > 1$ one sets

$$\widehat{S_\theta v}(\xi) = \chi(\xi/\theta) \hat{v}(\xi).$$

Then, if $s \geq t$,

$$\begin{aligned} (1 + |\xi|^2)^s |\widehat{S_\theta v}(\xi)|^2 &\leq \theta^{2(s-t)} (1 + |\xi/\theta|^2)^{s-t} |\chi(\xi/\theta)|^2 (1 + |\xi|^2)^t |\hat{v}(\xi)|^2 \\ &\leq (2\theta)^{2(s-t)} (1 + |\xi|^2)^t |\hat{v}(\xi)|^2 \end{aligned}$$

since $|\chi| \leq 1$ and $|\xi/\theta| \leq \sqrt{3}$ for $(\xi/\theta) \in \text{supp } \chi$; this gives the first estimate (4) with $C_{s,t} = 2^{s-t}$.

Similarly, for $s \leq t$,

$$(1 + |\xi|^2)^s |\hat{v}(\xi) - \widehat{S_\theta v}(\xi)|^2 = |1 - \chi(\xi/\theta)|^2 (1 + |\xi|^2)^s |\hat{v}(\xi)|^2;$$

a Taylor formula gives $|1 - \chi(\xi/\theta)| \leq C_k |\xi/\theta|^k$ with $C_k = \sup |\chi^{(k)}|/k!$ for any $k \in \mathbb{N}$ since $\chi(0) = 1$ and $\chi^{(j)}(0) = 0$ for $j > 0$, so that for $t = s + k$

$$\begin{aligned} (1 + |\xi|^2)^s |\hat{v}(\xi) - \widehat{S_\theta v}(\xi)|^2 &\leq C_{t-s}^2 |\xi/\theta|^{2(t-s)} (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 \\ &\leq C_{t-s}^2 \theta^{2(s-t)} (1 + |\xi|^2)^t |\hat{v}(\xi)|^2 \end{aligned}$$

whence the second estimate (4) with $C_{s,t} = C_{t-s} = \sup |\chi^{(t-s)}|/(t-s)!$

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