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If $\phi \in H^\infty(\Omega)$ is valued in $\mathbf{R}^{n(n+1)/2}$, let us consider it as a function valued in $\mathbf{R}^{n(n+3)/2}$ by adding n components $\varphi_j = 0$ for $1 \leq j \leq n$, and define $\psi(u)\phi$ as a continuous extension to \mathbf{R}^n of the function

$$(16) \quad v = -\frac{1}{2} A(u)^{-1} \phi$$

where $A(u)$ is the $n(n+3)/2$ square matrix the rows of which are $\partial_j u$ for $1 \leq j \leq n$ and $\partial_j \partial_k u$ for $1 \leq j \leq k \leq n$; thanks to our choice of u_0 , the matrix $A(u_0)$ is invertible on Ω , and so is $A(u)$ for any u close enough to u_0 . Since $A(u)^{-1}$ is an algebraic function of derivatives of u up to order 2, estimates such as (3) are again classical.

Finally, we have to prove that this operator ψ inverts ϕ' (formula (2)). Applying $A(u)$ to the function v in (16), one gets

$$\begin{aligned} \langle \partial_j u, v \rangle &= -\frac{1}{2} \varphi_j = 0 \quad 1 \leq j \leq n \\ \langle \partial_j \partial_k u, v \rangle &= -\frac{1}{2} \varphi_{jk} \quad 1 \leq j \leq k \leq n. \end{aligned}$$

The x_k derivative of the first equation gives $\langle \partial_j \partial_k u, v \rangle + \langle \partial_j u, \partial_k v \rangle = 0$, and one gets also $\langle \partial_j \partial_k u, v \rangle + \langle \partial_k u, \partial_j v \rangle = 0$ so that the second equation and (15) give $\phi'(u)v = \phi$ in Ω .

Thus all the assumptions of the theorem are fulfilled, and it follows that we can get a solution if $\phi(u_0)$ is sufficiently small in some $H^s(\Omega)$ norm; but according to (14), $\phi(u_0) = g^0 - g$, and the result is that (13) can be solved for any metric g close enough to g^0 , as required.

APPENDIX:

CONSTRUCTION OF THE SMOOTHING OPERATORS IN SOBOLEV SPACES

Let us recall that $v \in H^s(\mathbf{R}^n)$ means $v \in \mathcal{S}'(\mathbf{R}^n)$ and

$$|v|_s^2 = (2\pi)^{-n} \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty.$$

Let $\chi: \mathbf{R}^n \rightarrow [0, 1]$ be a C^∞ function taking the value 1 in a neighborhood of 0 and vanishing for $|\xi| \geq \sqrt{3}$. For $v \in H^\infty(\mathbf{R}^n)$ and $\theta > 1$ one sets

$$\widehat{S}_\theta v(\xi) = \chi(\xi/\theta) \hat{v}(\xi).$$

Then, if $s \geq t$,

$$\begin{aligned} (1 + |\xi|^2)^s |\widehat{S}_\theta v(\xi)|^2 &\leq \theta^{2(s-t)} (1 + |\xi/\theta|^2)^{s-t} |\chi(\xi/\theta)|^2 (1 + |\xi|^2)^t |\hat{v}(\xi)|^2 \\ &\leq (2\theta)^{2(s-t)} (1 + |\xi|^2)^t |\hat{v}(\xi)|^2 \end{aligned}$$

since $|\chi| \leq 1$ and $|\xi/\theta| \leq \sqrt{3}$ for $(\xi/\theta) \in \text{supp } \chi$; this gives the first estimate (4) with $C_{s,t} = 2^{s-t}$.

Similarly, for $s \leq t$,

$$(1+|\xi|^2)^s |\hat{v}(\xi)|^2 - \widehat{S_\theta v}(\xi) |^2 = |1 - \chi(\xi/\theta)|^2 (1+|\xi|^2)^s |\hat{v}(\xi)|^2;$$

a Taylor formula gives $|1 - \chi(\xi/\theta)| \leq C_k |\xi/\theta|^k$ with $C_k = \sup |\chi^{(k)}|/k!$ for any $k \in \mathbb{N}$ since $\chi(0) = 1$ and $\chi^{(j)}(0) = 0$ for $j > 0$, so that for $t = s + k$

$$\begin{aligned} (1+|\xi|^2)^s |\hat{v}(\xi)|^2 - \widehat{S_\theta v}(\xi) |^2 &\leq C_{t-s}^2 |\xi/\theta|^{2(t-s)} (1+|\xi|^2)^s |\hat{v}(\xi)|^2 \\ &\leq C_{t-s}^2 \theta^{2(s-t)} (1+|\xi|^2)^t |\hat{v}(\xi)|^2 \end{aligned}$$

whence the second estimate (4) with $C_{s,t} = C_{t-s} = \sup |\chi^{(t-s)}|/(t-s)!$

REFERENCES

- [1] HAMILTON, R. The inverse function theorem of Nash-Moser. *Bulletin of the A.M.S.* 7 (1982), 65-222.
- [2] HÖRMANDER, L. The boundary problems of physical geodesy. *Arch. Rat. Mech. Anal.* 62 (1976), 1-52.
- [3] —— *Implicit function theorems*. Lectures at Stanford University, Summer Quarter 1977.
- [4] —— On the Nash-Moser implicit function theorem. *Annales Acad. Sci. Fenniae, Series A.I. Math.* 10 (1985), 255-259.
- [5] MOSER, J. A new technique for the construction of solutions of nonlinear differential equations. *Proc. Nat. Acad. Sci.* 47 (1961), 1824-1831.
- [6] —— A rapidly convergent iteration method and nonlinear partial differential equations I and II. *Ann. Scuola Norm. Sup. di Pisa* 20 (1966), 265-315 and 499-533.
- [7] NASH, J. The imbedding problem for Riemannian manifolds. *Ann. of Math.* 63 (1956), 20-63.
- [8] SCHWARTZ, J. T. *Nonlinear functional analysis, Chap. II.A.* Gordon & Breach, New York 1969.
- [9] SERGERAERT, F. Une généralisation du théorème des fonctions implicites de Nash. *C. R. Acad. Sci. Paris*, 270A (1970), 861-863.
- [10] —— Un théorème des fonctions implicites sur certains espaces de Fréchet et quelques applications. *Ann. Sci. Ec. Norm. Sup. Paris 4^e série*, 5 (1972), 599-660.
- [11] ZEHNDER, E. Generalized implicit function theorems with applications to some small divisor problems I and II. *Comm. in Pure and Appl. Math.* 28 (1975), 91-140; 29 (1976), 49-111.

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