Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 35 (1989)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SIMPLE NASH-MOSER IMPLICIT FUNCTION THEOREM

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Anhang: Appendix : Construction of the Smoothing Operators in Sobolev Spaces

DOI: https://doi.org/10.5169/seals-57374

If $\phi \in H^\infty(\Omega)$ is valued in $\mathbf{R}^{n(n+1)/2}$, let us consider it as a function valued in $\mathbf{R}^{n(n+3)/2}$ by adding n components $\varphi_j = 0$ for $1 \leqslant j \leqslant n$, and define $\psi(u)\varphi$ as a continuous extension to $\mathbf{R}^n$ of the function

$$v = -\frac{1}{2} A(u)^{-1}\varphi \tag{16}$$

where $A(u)$ is the $n(n+3)/2$ square matrix the rows of which are $\partial_j u$ for $1 \leqslant j \leqslant n$ and $\partial_j \partial_k u$ for $1 \leqslant j \leqslant k \leqslant n$; thanks to our choice of u_0, the matrix $A(u_0)$ is invertible on Ω, and so is $A(u)$ for any u close enough to u_0. Since $A(u)^{-1}$ is an algebraic function of derivatives of u up to order 2, estimates such as (3) are again classical.

Finally, we have to prove that this operator ψ inverts ϕ' (formula (2)). Applying $A(u)$ to the function v in (16), one gets

$$\langle \partial_j u, v\rangle = -\frac{1}{2}\varphi_j = 0 \qquad 1 \leqslant j \leqslant n$$

$$\langle \partial_j \partial_k u, v\rangle = -\frac{1}{2}\varphi_{jk} \qquad 1 \leqslant j \leqslant k \leqslant n\,.$$

The x_k derivative of the first equation gives $\langle \partial_j \partial_k u, v\rangle + \langle \partial_j u, \partial_k v\rangle = 0$, and one gets also $\langle \partial_j \partial_k u, v\rangle + \langle \partial_k u, \partial_j v\rangle = 0$ so that the second equation and (15) give $\phi'(u)v = \varphi$ in Ω.

Thus all the assumptions of the theorem are fulfilled, and it follows that we can get a solution if $\phi(u_0)$ is sufficiently small in some $H^s(\Omega)$ norm; but according to (14), $\phi(u_0) = g^0 - g$, and the result is that (13) can be solved for any metric g close enough to g^0, as required.

APPENDIX:

CONSTRUCTION OF THE SMOOTHING OPERATORS IN SOBOLEV SPACES

Let us recall that $v \in H^s(\mathbf{R}^n)$ means $v \in \mathcal{S}'(\mathbf{R}^n)$ and

$$|v|_s^2 = (2\pi)^{-n}\int (1+|\xi|^2)^s\,|\hat{u}(\xi)|^2 d\xi < \infty\,.$$

Let $\chi\colon \mathbf{R}^n \to [0, 1]$ be a C^∞ function taking the value 1 in a neighborhood of 0 and vanishing for $|\xi| \geqslant \sqrt{3}$. For $v \in H^\infty(\mathbf{R}^n)$ and $\theta > 1$ one sets

$$\widehat{S_\theta v}(\xi) = \chi(\xi/\theta)\hat{v}(\xi)\,.$$

Then, if $s \geqslant t$,

$$\begin{aligned}(1+|\xi|^2)^s\,|\widehat{S_\theta v}(\xi)|^2 &\leqslant \theta^{2(s-t)}(1+|\xi/\theta|^2)^{s-t}\,|\chi(\xi/\theta)|^2(1+|\xi|^2)^t\,|\hat{v}(\xi)|^2\\ &\leqslant (2\theta)^{2(s-t)}(1+|\xi|^2)^t\,|\hat{v}(\xi)|^2\end{aligned}$$

since $|\chi| \leqslant 1$ and $|\xi/\theta| \leqslant \sqrt{3}$ for $(\xi/\theta) \in \operatorname{supp} \chi$; this gives the first estimate (4) with $C_{s,t} = 2^{s-t}$.

Similarly, for $s \leqslant t$,

$$(1+|\xi|^2)^s \,|\, \hat{v}(\xi) - \widehat{S_\theta v}(\xi) \,|^2 = |\, 1 - \chi(\xi/\theta) \,|^2 (1+|\xi|^2)^s \,|\, \hat{v}(\xi) \,|^2 ;$$

a Taylor formula gives $|\, 1 - \chi(\xi/\theta) \,| \leqslant C_k \,|\, \xi/\theta \,|^k$ with $C_k = \sup |\, \chi^{(k)} \,|/k!$ for any $k \in \mathbf{N}$ since $\chi(0) = 1$ and $\chi^{(j)}(0) = 0$ for $j > 0$, so that for $t = s + k$

$$\begin{aligned}(1+|\xi|^2)^s \,|\, \hat{v}(\xi) - \widehat{S_\theta v}(\xi) \,|^2 &\leqslant C_{t-s}^2 \,|\, \xi/\theta \,|^{2(t-s)} (1+|\xi|^2)^s \,|\, \hat{v}(\xi) \,|^2 \\ &\leqslant C_{t-s}^2 \theta^{2(s-t)} (1+|\xi|^2)^t \,|\, \hat{v}(\xi) \,|^2\end{aligned}$$

whence the second estimate (4) with $C_{s,t} = C_{t-s} = \sup |\, \chi^{(t-s)} \,|/(t-s)!$

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(Reçu le 14 juin 1989)

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