

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 35 (1989)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SIMPLE NASH-MOSER IMPLICIT FUNCTION THEOREM
Autor: Raymond, Xavier Saint
Kapitel: Application to the local isometric embedding of a Riemannian manifold
DOI: <https://doi.org/10.5169/seals-57374>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 17.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

APPLICATION TO THE LOCAL ISOMETRIC EMBEDDING
OF A RIEMANNIAN MANIFOLD

(following Hörmander [3], Section 2).

Let M be a compact C^∞ manifold of dimension n and g a smooth Riemannian metric on M . In local coordinates, we are thus given a positive definite quadratic form

$$g = \sum_{j,k} g_{jk} dx_j dx_k.$$

The celebrated theorem of Nash [7], which is at the origin of the method, states that for some (large) integer N , there is an isometric embedding $u: M \rightarrow \mathbf{R}^N$, that is an injective map satisfying the system of equations

$$(13) \quad \langle \partial_j u, \partial_k u \rangle = g_{jk} \quad 1 \leq j, k \leq n$$

where ∂_j stands for $\partial/\partial x_j$ and $\langle \cdot, \cdot \rangle$ for the Euclidean scalar product in \mathbf{R}^N ; thus, any compact Riemannian manifold can be thought as a submanifold of a Euclidean space.

In the proof of this Nash theorem, one first establishes that the set of metrics g such that the problem can be solved is a dense convex cone in the set of all C^∞ metrics on M , and this leads to the following reduced problem (see Hörmander [3] Section 2): show that the equation (13) can be solved for every metric in some neighborhood of a fixed metric g^0 .

To illustrate the method described above, let us show how one can use our theorem to prove this last property locally (and this will give a local isometric embedding $u: M \rightarrow \mathbf{R}^N$).

Let $\Omega = \{x \in \mathbf{R}^n; |x| < 1\}$ and choose, near some point $x_0 \in M$, local coordinates such that Ω describes a neighborhood of x_0 ; we take a $C_0^\infty u_0: \mathbf{R}^n \rightarrow \mathbf{R}^{n(n+3)/2}$ equal to

$$((x_j)_{1 \leq j \leq n}, (x_j^2/2)_{1 \leq j \leq n}, (x_j x_k)_{1 \leq j < k \leq n})$$

in a neighborhood of $\bar{\Omega}$; this u_0 is an isometric embedding for the corresponding metric g^0 in Ω , namely the metric $g_{jj}^0 = 1 + |x|^2$ and $g_{jk} = x_j x_k$ if $j \neq k$. Finally, for a metric g close to g^0 , we consider the restriction $\phi(u)$ to Ω of the function

$$(14) \quad (\langle \partial_j u, \partial_k u \rangle - g_{jk})_{1 \leq j \leq k \leq n}$$

which is a function in $H^\infty(\Omega)$ valued in $\mathbf{R}^{n(n+1)/2}$ for any $u \in H^\infty(\mathbf{R}^n)$ valued in $\mathbf{R}^{n(n+3)/2}$. Classically, estimates such as (1) hold for $s > (n+2)/2$.

The derivative of ϕ with respect to u is defined by

$$(15) \quad \phi'(u)v = (\langle \partial_j u, \partial_k v \rangle + \langle \partial_k u, \partial_j v \rangle)_{1 \leq j \leq k \leq n}.$$

If $\phi \in H^\infty(\Omega)$ is valued in $\mathbf{R}^{n(n+1)/2}$, let us consider it as a function valued in $\mathbf{R}^{n(n+3)/2}$ by adding n components $\phi_j = 0$ for $1 \leq j \leq n$, and define $\psi(u)\phi$ as a continuous extension to \mathbf{R}^n of the function

$$(16) \quad v = -\frac{1}{2} A(u)^{-1} \phi$$

where $A(u)$ is the $n(n+3)/2$ square matrix the rows of which are $\partial_j u$ for $1 \leq j \leq n$ and $\partial_j \partial_k u$ for $1 \leq j \leq k \leq n$; thanks to our choice of u_0 , the matrix $A(u_0)$ is invertible on Ω , and so is $A(u)$ for any u close enough to u_0 . Since $A(u)^{-1}$ is an algebraic function of derivatives of u up to order 2, estimates such as (3) are again classical.

Finally, we have to prove that this operator ψ inverts ϕ' (formula (2)). Applying $A(u)$ to the function v in (16), one gets

$$\begin{aligned} \langle \partial_j u, v \rangle &= -\frac{1}{2} \phi_j = 0 & 1 \leq j \leq n \\ \langle \partial_j \partial_k u, v \rangle &= -\frac{1}{2} \phi_{jk} & 1 \leq j \leq k \leq n. \end{aligned}$$

The x_k derivative of the first equation gives $\langle \partial_j \partial_k u, v \rangle + \langle \partial_j u, \partial_k v \rangle = 0$, and one gets also $\langle \partial_j \partial_k u, v \rangle + \langle \partial_k u, \partial_j v \rangle = 0$ so that the second equation and (15) give $\phi'(u)v = \phi$ in Ω .

Thus all the assumptions of the theorem are fulfilled, and it follows that we can get a solution if $\phi(u_0)$ is sufficiently small in some $H^s(\Omega)$ norm; but according to (14), $\phi(u_0) = g^0 - g$, and the result is that (13) can be solved for any metric g close enough to g^0 , as required.

APPENDIX:

CONSTRUCTION OF THE SMOOTHING OPERATORS IN SOBOLEV SPACES

Let us recall that $v \in H^s(\mathbf{R}^n)$ means $v \in \mathcal{S}'(\mathbf{R}^n)$ and

$$|v|_s^2 = (2\pi)^{-n} \int (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 d\xi < \infty.$$

Let $\chi: \mathbf{R}^n \rightarrow [0, 1]$ be a C^∞ function taking the value 1 in a neighborhood of 0 and vanishing for $|\xi| \geq \sqrt{3}$. For $v \in H^\infty(\mathbf{R}^n)$ and $\theta > 1$ one sets

$$\widehat{S_\theta v}(\xi) = \chi(\xi/\theta) \hat{v}(\xi).$$

Then, if $s \geq t$,

$$\begin{aligned} (1 + |\xi|^2)^s |\widehat{S_\theta v}(\xi)|^2 &\leq \theta^{2(s-t)} (1 + |\xi/\theta|^2)^{s-t} |\chi(\xi/\theta)|^2 (1 + |\xi|^2)^t |\hat{v}(\xi)|^2 \\ &\leq (2\theta)^{2(s-t)} (1 + |\xi|^2)^t |\hat{v}(\xi)|^2 \end{aligned}$$