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APPLICATION TO THE LOCAL ISOMETRIC EMBEDDING
OF A RIEMANNIAN MANIFOLD

(following Hörmander [3], Section 2).

Let M be a compact C^∞ manifold of dimension n and g a smooth Riemannian metric on M . In local coordinates, we are thus given a positive definite quadratic form

$$g = \sum_{j,k} g_{jk} dx_j dx_k.$$

The celebrated theorem of Nash [7], which is at the origin of the method, states that for some (large) integer N , there is an isometric embedding $u: M \rightarrow \mathbf{R}^N$, that is an injective map satisfying the system of equations

$$(13) \quad \langle \partial_j u, \partial_k u \rangle = g_{jk} \quad 1 \leq j, k \leq n$$

where ∂_j stands for $\partial/\partial x_j$ and $\langle \cdot, \cdot \rangle$ for the Euclidean scalar product in \mathbf{R}^N ; thus, any compact Riemannian manifold can be thought as a submanifold of a Euclidean space.

In the proof of this Nash theorem, one first establishes that the set of metrics g such that the problem can be solved is a dense convex cone in the set of all C^∞ metrics on M , and this leads to the following reduced problem (see Hörmander [3] Section 2): show that the equation (13) can be solved for every metric in some neighborhood of a fixed metric g^0 .

To illustrate the method described above, let us show how one can use our theorem to prove this last property locally (and this will give a local isometric embedding $u: M \rightarrow \mathbf{R}^N$).

Let $\Omega = \{x \in \mathbf{R}^n; |x| < 1\}$ and choose, near some point $x_0 \in M$, local coordinates such that Ω describes a neighborhood of x_0 ; we take a $C_0^\infty u_0: \mathbf{R}^n \rightarrow \mathbf{R}^{n(n+3)/2}$ equal to

$$((x_j)_{1 \leq j \leq n}, (x_j^2/2)_{1 \leq j \leq n}, (x_j x_k)_{1 \leq j < k \leq n})$$

in a neighborhood of $\bar{\Omega}$; this u_0 is an isometric embedding for the corresponding metric g^0 in Ω , namely the metric $g_{jj}^0 = 1 + |x|^2$ and $g_{jk} = x_j x_k$ if $j \neq k$. Finally, for a metric g close to g^0 , we consider the restriction $\phi(u)$ to Ω of the function

$$(14) \quad (\langle \partial_j u, \partial_k u \rangle - g_{jk})_{1 \leq j \leq k \leq n}$$

which is a function in $H^\infty(\Omega)$ valued in $\mathbf{R}^{n(n+1)/2}$ for any $u \in H^\infty(\mathbf{R}^n)$ valued in $\mathbf{R}^{n(n+3)/2}$. Classically, estimates such as (1) hold for $s > (n+2)/2$.

The derivative of ϕ with respect to u is defined by

$$(15) \quad \phi'(u)v = (\langle \partial_j u, \partial_k v \rangle + \langle \partial_k u, \partial_j v \rangle)_{1 \leq j \leq k \leq n}.$$

If $\phi \in H^\infty(\Omega)$ is valued in $\mathbf{R}^{n(n+1)/2}$, let us consider it as a function valued in $\mathbf{R}^{n(n+3)/2}$ by adding n components $\phi_j = 0$ for $1 \leq j \leq n$, and define $\psi(u)\phi$ as a continuous extension to \mathbf{R}^n of the function

$$(16) \quad v = -\frac{1}{2} A(u)^{-1} \phi$$

where $A(u)$ is the $n(n+3)/2$ square matrix the rows of which are $\partial_j u$ for $1 \leq j \leq n$ and $\partial_j \partial_k u$ for $1 \leq j \leq k \leq n$; thanks to our choice of u_0 , the matrix $A(u_0)$ is invertible on Ω , and so is $A(u)$ for any u close enough to u_0 . Since $A(u)^{-1}$ is an algebraic function of derivatives of u up to order 2, estimates such as (3) are again classical.

Finally, we have to prove that this operator ψ inverts ϕ' (formula (2)). Applying $A(u)$ to the function v in (16), one gets

$$\begin{aligned} \langle \partial_j u, v \rangle &= -\frac{1}{2} \phi_j = 0 & 1 \leq j \leq n \\ \langle \partial_j \partial_k u, v \rangle &= -\frac{1}{2} \phi_{jk} & 1 \leq j \leq k \leq n. \end{aligned}$$

The x_k derivative of the first equation gives $\langle \partial_j \partial_k u, v \rangle + \langle \partial_j u, \partial_k v \rangle = 0$, and one gets also $\langle \partial_j \partial_k u, v \rangle + \langle \partial_k u, \partial_j v \rangle = 0$ so that the second equation and (15) give $\phi'(u)v = \phi$ in Ω .

Thus all the assumptions of the theorem are fulfilled, and it follows that we can get a solution if $\phi(u_0)$ is sufficiently small in some $H^s(\Omega)$ norm; but according to (14), $\phi(u_0) = g^0 - g$, and the result is that (13) can be solved for any metric g close enough to g^0 , as required.

APPENDIX:

CONSTRUCTION OF THE SMOOTHING OPERATORS IN SOBOLEV SPACES

Let us recall that $v \in H^s(\mathbf{R}^n)$ means $v \in \mathcal{S}'(\mathbf{R}^n)$ and

$$|v|_s^2 = (2\pi)^{-n} \int (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 d\xi < \infty.$$

Let $\chi: \mathbf{R}^n \rightarrow [0, 1]$ be a C^∞ function taking the value 1 in a neighborhood of 0 and vanishing for $|\xi| \geq \sqrt{3}$. For $v \in H^\infty(\mathbf{R}^n)$ and $\theta > 1$ one sets

$$\widehat{S_\theta v}(\xi) = \chi(\xi/\theta) \hat{v}(\xi).$$

Then, if $s \geq t$,

$$\begin{aligned} (1 + |\xi|^2)^s |\widehat{S_\theta v}(\xi)|^2 &\leq \theta^{2(s-t)} (1 + |\xi/\theta|^2)^{s-t} |\chi(\xi/\theta)|^2 (1 + |\xi|^2)^t |\hat{v}(\xi)|^2 \\ &\leq (2\theta)^{2(s-t)} (1 + |\xi|^2)^t |\hat{v}(\xi)|^2 \end{aligned}$$