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In general this result does not hold for infinite semigroups. We invite the reader to find a counterexample for  $S = (\mathbb{Z}, +)$ . However it does hold for compact semigroups as we will show implicitly in the proof of Theorem 3.3. (Of course the finite version is then a special case.)

We intend to apply this theorem to the natural numbers  $\mathbb{N}$  by compactifying  $\mathbb{N}$  in such a way so as to obtain a compact semigroup; this is the role of the Stone-Čech compactification  $\beta\mathbb{N}$  of  $\mathbb{N}$ . We obtain a theorem about  $\beta\mathbb{N}$  which when unraveled becomes exactly van der Waerden's Theorem.

We warn the reader that in the compactification of  $\mathbb{N}$  the operation of addition will be extended with the usual notation  $+$ . However the semigroup will not be commutative and so one has to accustom oneself to non-commutative addition.

## 1. SEMIGROUP PROPERTIES OF $\beta\mathbb{N}$

Any completely regular Hausdorff space has a maximal compactification, the Stone-Čech compactification. In particular the discrete space  $\mathbb{N}$  of positive integers has a Stone-Čech compactification  $\beta\mathbb{N}$  which is characterized by: (1)  $\beta\mathbb{N}$  is a compact Hausdorff space; (2)  $\mathbb{N}$  is a dense subset of  $\beta\mathbb{N}$ ; and (3) given any compact Hausdorff space  $Y$  and any  $f: \mathbb{N} \rightarrow Y$  there is a continuous extension  $f^\beta: \beta\mathbb{N} \rightarrow Y$ , (that is  $f^\beta|_{\mathbb{N}} = f$ ).

Our proof of van der Waerden's Theorem is based on the fact that the operation of ordinary addition extends to  $\beta\mathbb{N}$  as an operation which we denote by  $+$ .  $\beta\mathbb{N}$  under this operation will be a semigroup in which the operation of addition is continuous in a restricted way. Namely let  $(S, +)$  be a semigroup with  $S$  a topological space and define functions  $\rho_x$  and  $\lambda_x$  for each  $x \in S$  by  $\rho_x(y) = y + x$  and  $\lambda_x(y) = x + y$ . If one requires only that  $\rho_x$  be continuous,  $S$  is called a right topological semigroup.

**1.1 LEMMA.** *There is an operation  $+$  on  $\beta\mathbb{N}$  such that  $\beta\mathbb{N}$  is a compact right topological semigroup,  $+$  extends ordinary addition on  $\mathbb{N}$ , and  $\lambda_n$  is continuous for each  $n \in \mathbb{N}$ .*

*Proof.* We extend  $+$  in stages, starting with  $+$  defined on  $\mathbb{N} \times \mathbb{N}$ . Given  $n \in \mathbb{N}$ , consider  $f_n: \mathbb{N} \rightarrow \beta\mathbb{N}$  defined by  $f_n(m) = n + m$ . Then each  $f_n$  has a continuous extension  $f_n^\beta: \beta\mathbb{N} \rightarrow \beta\mathbb{N}$ . For  $n \in \mathbb{N}$  and  $p \in \beta\mathbb{N} \setminus \mathbb{N}$  define

$n + p = f_n^\beta(p)$ . (Then for  $n \in \mathbf{N}$  and any  $p \in \beta\mathbf{N}$ ,  $n + p = f_n^\beta(p)$  since if  $p \in \mathbf{N}$ ,  $f_n^\beta(p) = f_n(p) = n + p$ .) Now  $+$  is defined on  $\mathbf{N} \times \beta\mathbf{N}$ . Given  $p \in \beta\mathbf{N}$  define  $g_p: \mathbf{N} \rightarrow \beta\mathbf{N}$  by  $g_p(n) = n + p$ . Then each  $g_p$  has a continuous extension  $g_p^\beta: \beta\mathbf{N} \rightarrow \beta\mathbf{N}$ . Then for  $p \in \beta\mathbf{N}$  and  $q \in \beta\mathbf{N} \setminus \mathbf{N}$  define  $q + p = g_p^\beta(q)$ . (Again if  $p, q$  are any points in  $\beta\mathbf{N}$  we have  $q + p = g_p^\beta(q)$ .)

Since for any  $n \in \mathbf{N}$ ,  $\lambda_n = f_n^\beta$  and for any  $p \in \beta\mathbf{N}$ ,  $\rho_p = g_p^\beta$ , the continuity assumptions are immediate. Thus we need only check that the operation is associative. To this end let  $p, q, r \in \beta\mathbf{N}$ . Observe that  $p + (q + r) = \rho_{q+r}(p)$  while  $(p + q) + r = (\rho_r \circ \rho_q)(p)$  so by continuity it suffices to show  $\rho_{q+r}$  and  $\rho_r \circ \rho_q$  agree on the dense subset  $\mathbf{N}$  of  $\beta\mathbf{N}$ . Let  $n \in \mathbf{N}$ . Then

$$\rho_{q+r}(n) = n + (q + r) = (\lambda_n \circ \rho_r)(q)$$

$$\text{and } (\rho_r \circ \rho_q)(n) = (n + q) + r = (\rho_r \circ \lambda_n)(q).$$

Again by continuity, it suffices to show  $\lambda_n \circ \rho_r$  and  $\rho_r \circ \lambda_n$  agree on  $\mathbf{N}$ . Let  $m \in \mathbf{N}$ . Then

$$(\lambda_n \circ \rho_r)(m) = n + (m + r) = (\lambda_n \circ \lambda_m)(r)$$

while

$$(\rho_r \circ \lambda_n)(m) = (n + m) + r = \lambda_{n+m}(r).$$

Thus it finally suffices to show  $\lambda_n \circ \lambda_m$  and  $\lambda_{n+m}$  agree on  $\mathbf{N}$ . Let  $t \in \mathbf{N}$ . Then  $(\lambda_n \circ \lambda_m)(t) = n + (m + t) = (n + m) + t = \lambda_{n+m}(t)$  as required.  $\square$

The main fact about  $\beta\mathbf{N}$  making it useful for van der Waerden's Theorem and similar results is the content of the following lemma.

**1.2 LEMMA.** *If  $\{A_1, A_2, \dots, A_m\}$  is a finite partition of  $\mathbf{N}$ , then  $\{cl A_1, cl A_2, \dots, cl A_m\}$  is a partition of  $\beta\mathbf{N}$  such that for each  $i \in \{1, 2, \dots, m\}$ ,  $cl A_i$  is open.*

*Proof.* Let  $Y = \{1, 2, \dots, m\}$  with the discrete topology and define  $f: \mathbf{N} \rightarrow Y$  by  $f(n) = i$  if and only if  $n \in A_i$ . For each  $i \in \{1, 2, \dots, m\}$ , let  $B_i = \{p \in \beta\mathbf{N}: f^\beta(p) = i\}$ . Then immediately  $\{B_1, B_2, \dots, B_m\}$  is a partition of  $\beta\mathbf{N}$ . Further, given  $i \in \{1, 2, \dots, m\}$ ,  $B_i = (f^\beta)^{-1}[\{i\}]$ . Since  $\{i\}$  is open and closed in  $Y$  and  $f^\beta$  is continuous,  $B_i$  is open and closed. Since  $A_i \subseteq B_i$ , one has  $cl A_i \subseteq B_i$ . To see that  $B_i \subseteq cl A_i$ , let  $x \in B_i$  and let  $U$  be a neighborhood of  $x$ . Since  $\mathbf{X}$  is dense in  $\beta\mathbf{N}$ , pick  $y \in \mathbf{N} \cap (U \cap B_i)$ . Since  $y \in B_i$ ,  $f(y) = i$  so  $y \in A_i$ . Thus  $U \cap A_i \neq \emptyset$  as required.  $\square$